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FOURIER INVERSION FOR UNIPOTENT INVARIANT INTEGRALS

BY

DAN BARBASCH[1]

ABSTRACT. Consider G a semisimple Lie group and $\Gamma \subseteq G$ a discrete subgroup such that $\mathrm{vol}(G/\Gamma) < \infty$. An important problem for number theory and representation theory is to find the decomposition of $L^2(G/\Gamma)$ into irreducible representations. Some progress in this direction has been made by J. Arthur and G. Warner by using the Selberg trace formula, which expresses the trace of a subrepresentation of $L^2(G/\Gamma)$ in terms of certain invariant distributions. In particular, measures supported on orbits of unipotent elements of G occur. In order to obtain information about representations it is necessary to expand these distributions into Fourier components using characters of irreducible unitary representations of G. This problem is solved in this paper for real rank $G = 1$. In particular, a relationship between the semisimple orbits and the nilpotent ones is made explicit generalizing an earlier result of R. Rao.

1. Introduction. Let G be a connected semisimple Lie group with finite center and Lie algebra $\mathfrak{g}$. Let q be any element of G and $O(q)$ be the orbit of q under the adjoint action of G. Then $O(q)$ carries a G-invariant measure which we denote by T_q. From the results in **[11b]** it follows that T_q has the important property that it defines a tempered distribution on G and is also a measure on G. Then it makes sense to try to obtain Fourier inversion formulas in the sense of Harish-Chandra for the distributions T_q. The problem considered in this paper is to find such explicit formulas. This means to express the distribution T_q in terms of a series of tempered invariant eigendistributions of G. These series include the characters of the discrete series representations (if G has a compact Cartan subgroup), characters of the unitary principal series and certain eigendistributions that can be interpreted as alternating sums of characters **[7]**.

Such formulas have been obtained in **[12]** for q semisimple and G of real rank 1. For q regular and G of rank higher than 1, formulas are obtained in **[6]**.

In this paper we derive an inversion formula for q unipotent in case G has real rank one. These distributions occur in the Selberg trace formula derived in **[10]**. The inversion formulas derived here are useful in obtaining

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information about the multiplicities of irreducible representations occurring in $L_0^2(\Gamma\setminus G)$, where Γ is a discrete subgroup of G with $\mathrm{vol}(\Gamma\setminus G) < \infty$.

The method used for obtaining the formulas is an extension of some unpublished results of R. Rao **[11a]**. Let u be a unipotent element, $u = \exp X$ and $\{X, H, Y\}$ be a Lie triple. Then let $\mathfrak{U} = X + Z_Y$ where $Z_Y = \mathrm{Cent}_{\mathfrak{g}} Y$. In **[11a]** it is shown that $\mathfrak{U}$ is a transverse to the adjoint action of G on the Lie algebra $\mathfrak{g}$. Since both X and $X - Y$ are elements of $\mathfrak{U}$ and $\bigcup_{t>0} O(t(X - Y))$ contains the orbit of X in its closure, the formula in Theorem 6.7 can be proved for a general class of nilpotents (such that ad H has even eigenvalues only) by lifting the integrals to the transverse $\mathfrak{U}$. The main difficulties in using this method for an arbitrary nilpotent are the fact that $\mathfrak{U}^G$ is not equal to the entire Lie algebra $\mathfrak{g}$, the fact that there may be several conjugacy classes of nilpotents conjugate under G_c but not under the real group G and finally convergence problems on $\mathfrak{U}$.

In order to avoid these difficulties we take a slightly different approach. In §§4 and 5 we construct transverses to the orbits of X and $Z = X - Y$ by using a decomposition theorem of Mostow **[9]** and lift the invariant measures T_X and T_Z to the corresponding transverses. Then it becomes apparent that we only need to consider the case su(2, 1). In §6 the relation between T_Z and T_X is calculated. The central result is Theorem 6.7. In §7 the Fourier inversion formula for T_u is derived by differentiation from the formulas in **[12]**, while in §8 the constants relevant to the results in **[10]** are computed explicitly. These results as well as the inversion for an arbitrary q are proved in a different way in **[2]**.

I am indebted to Professor R. Rao for many helpful suggestions.

2. Notation and preliminary results. Let $\mathfrak{g}$ be a real semisimple Lie algebra with complexification $\mathfrak{g}_c$. Let σ be the Lie conjugation of $\mathfrak{g}_c$ defined by $\mathfrak{g}$. Fix once and for all a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of $\mathfrak{g}$. Then $\mathfrak{k} + \sqrt{-1}\,\mathfrak{p}$ is a compact real form of $\mathfrak{g}_c$. Denote by τ the conjugation defined by this real form. Then τ and σ commute and $\tau\sigma$ is a Cartan involution of $\mathfrak{g}$ giving rise to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Choose a subspace $\mathfrak{a}$ and $\mathfrak{n}$ such that $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is the Iwasawa decomposition of $\mathfrak{g}$.

Let G be a connected semisimple Lie group with finite center and Lie algebra $\mathfrak{g}$. If G_c is the connected simply connected group corresponding to $\mathfrak{g}_c$, we assume that G is the real subgroup of G_c corresponding to $\mathfrak{g}$. We also denote by K the subgroup corresponding to $\mathfrak{k}$, by A the subgroup corresponding to $\mathfrak{a}$ and by N the subgroup corresponding to $\mathfrak{n}$.

Let $\mathfrak{j} \subseteq \mathfrak{g}$ be an Iwasawa Cartan subalgebra which is τ-invariant and has the property that $\mathfrak{j} \cap \mathfrak{p} = \mathfrak{a}$. Then there is a direct sum decomposition $\mathfrak{j} = \mathfrak{a}_K + \mathfrak{a}$ where $\mathfrak{a}_K = \mathfrak{j} \cap \mathfrak{k}$. If J is the Cartan subgroup with Lie algebra $\mathfrak{j}$, then $J = A \cdot A_K$ where $A = \exp \mathfrak{a}$ and $A_K = J \cap K$.

Denote by $\Delta(\mathfrak{g}_c, \mathfrak{j}_c)$ the root system of $\mathfrak{g}_c$ corresponding to $\mathfrak{j}_c$. We will drop the parentheses and write Δ whenever there is no danger of confusion. It is well known that if $\beta \in \Delta$ then $\sigma\beta \in \Delta$ where $\sigma\beta$ is defined by $(\sigma\beta)(H) = \overline{\beta(\sigma H)}$ (the bar denotes complex conjugation) for any $H \in \mathfrak{j}_c$.

We also denote by $\Delta(\mathfrak{g}, \mathfrak{a})$ the restricted root system and introduce compatible orderings on $\Delta(\mathfrak{g}_c, \mathfrak{j}_c)$ and $\Delta(\mathfrak{g}, \mathfrak{a})$ such that if $\beta > 0$ then $\sigma\beta > 0$ as well, provided $\beta + \sigma\beta \neq 0$. A root such that $\beta + \sigma\beta = 0$ is called imaginary and is characterized by the fact that $\beta|_{\mathfrak{a}} = 0$. We denote the set of imaginary roots by Δ_0. A root α is called real if $\alpha = \sigma\alpha$. It is characterized by the property that $\alpha|_{\mathfrak{a}_K} = 0$.

Let B be the Cartan-Killing form of $\mathfrak{g}_c$ and $B_\tau(X, Y) = B(X, \tau Y)$ be the associated negative definite form. The restriction of B to $\mathfrak{j}_c$ will be denoted by $(\, , \,)$. Since B is nondegenerate we can define a form on the dual of $\mathfrak{j}_c$ which will be denoted by $(\, , \,)$ as well. For $\beta \in \Delta$ define H_β and H'_β by the relations $(H, H_\beta) = \beta(H)$ for any H, and $H'_\beta = (2/(\beta, \beta))H_\beta$.

Let E_β be the root vector associated to the root β. Then $\sigma E_\beta = \rho_\beta E_{\sigma\beta}$ where $|\rho_\beta| = 1$. We use the Weyl normalization for these vectors so that $\tau E_\beta = -E_{-\beta}$. Then the vectors E_β, $E_{-\beta}$ and H'_β satisfy the relations

$$[H'_\beta, E_\beta] = 2E_\beta, \quad [H'_\beta, E_{-\beta}] = -2E_{-\beta}, \quad [E_\beta, E_{-\beta}] = -H'_\beta.$$

The following well-known facts are going to be used later.

LEMMA 2.1. *Let $\beta, \gamma \in \Delta$. Then there are integers $p, q \in \mathbf{N}$ such that $\gamma + s\beta$ is a root if and only if $-q \leq s \leq p$ and s is an integer. In addition $2(\beta, \gamma)/(\gamma, \gamma) = q - p$ and $\beta - 2(\beta, \gamma)\gamma/(\gamma, \gamma)$ is a root.*

PROOF. See [**13**, p. 41].

COROLLARY 2.2. *Let $\alpha, \beta \in \mathfrak{j}_c^*$, the dual of $\mathfrak{j}_c$. Then*

$$|(\alpha, \beta)|^2 \leq (\alpha, \alpha)(\beta, \beta).$$

If $\alpha, \beta \in \Delta$ then equality holds if and only if $\alpha = \pm\beta$.

PROOF. The first statement is the Cauchy-Schwartz inequality. The second is a consequence of the fact that the only multiples of β that are roots are $\pm\beta$.

LEMMA 2.3. *If β is a root then $\beta - \sigma\beta$ is not a root.*

PROOF. See [**1**, p. 6].

LEMMA 2.4. *Let β and γ be positive roots such that $\beta \neq \gamma$ and $\beta \neq \sigma\gamma$. If $\beta|_{\mathfrak{a}}$ and $\gamma|_{\mathfrak{a}}$ are multiples of each other then either $\beta - \gamma$ or $\beta - \sigma\gamma$ is a root.*

PROOF. We have the relation

$$(\beta, \gamma) + (\beta, \sigma\gamma) = (\beta, \gamma + \sigma\gamma) = (\beta, 2\gamma|_{\mathfrak{a}}) = 2(\beta|_{\mathfrak{a}}, \gamma|_{\mathfrak{a}}) > 0,$$

so either $(\beta, \gamma) > 0$ or $(\beta, \sigma\gamma) > 0$, say $(\beta, \gamma) > 0$. Then Lemma 2.1 implies that $\beta - \gamma$ must be a root and the proof is complete.

Next, we consider the conjugacy classes of Cartan subalgebras of $\mathfrak{g}$ for the case when $\mathrm{rank}(G/K) = 1$. It is well known that any Cartan subalgebra can be conjugated to a τ-invariant one. Let $\mathfrak{h}$ be such a subalgebra and $\mathfrak{h} = \mathfrak{h}_k + \mathfrak{h}_p$ be its Cartan decomposition. Since $\mathrm{rank}(G/K) = 1$ it follows that $\dim \mathfrak{h}_p \leqslant 1$. The Cartan subalgebras such that $\dim \mathfrak{h}_p = 1$ are all conjugate to the Iwasawa Cartan subalgebra $\mathfrak{j}$. The Cartan subalgebras such that $\dim \mathfrak{h}_p = 0$ are all conjugate. They are called fundamental or also compact Cartan subalgebras. The following result is well known. We include a proof for completeness.

PROPOSITION 2.5. *Let* $\mathfrak{g}$ *be a real semisimple algebra such that* $\mathrm{rank}(G/K) = 1$. *Then we have the following two cases.*

(1) *If* $\mathrm{rank}\ \mathfrak{g} > \mathrm{rank}\ \mathfrak{k}$ *then the fundamental and the Iwasawa Cartan subalgebra are the same. There is only one conjugacy class of Cartan subalgebras.*

(2) *If* $\mathrm{rank}\ \mathfrak{g} = \mathrm{rank}\ \mathfrak{k}$ *then there is a real root* $\alpha \in \Delta(\mathfrak{g}_c, \mathfrak{j}_c)$. *A representative for the fundamental Cartan subalgebra is* $\mathfrak{b} = \mathfrak{a}_K + \mathbf{R}(E_\alpha - E_{-\alpha})$. *The subalgebras* $\mathfrak{j}$ *and* $\mathfrak{b}$ *are not conjugate by* G *but the element* $\nu = \exp[\pi\sqrt{-1}\,(E_\alpha + E_{-\alpha})/4]$ *conjugates* $\mathfrak{j}_c$ *into* $\mathfrak{b}_c$.

PROOF. The first statement is straightforward. Assume $\mathrm{rank}\ \mathfrak{g} = \mathrm{rank}\ \mathfrak{k}$. Let $\mathfrak{b} \subseteq \mathfrak{k}$ be a Cartan subalgebra. Since not all roots of $\Delta(\mathfrak{g}_c, \mathfrak{b}_c)$ are compact, there is a singular imaginary root $\tilde{\alpha}$. Select $Z \in \mathfrak{b}$ such that the only roots vanishing at Z are $\pm\tilde{\alpha}$. Let c_Z be the center of $\mathrm{Cent}_{\mathfrak{g}} Z$ and l_Z be the derived algebra. Then l_Z is isomorphic to $\mathrm{sl}(2, \mathbf{R})$ and we can select a basis X^*, H^*, Y^* such that $\tau X^* = -Y^*$ and $[X^*, Y^*] = H^*$, $[H^*, X^*] = 2X^*$, $[H^*, Y^*] = -2Y^*$. Then $\mathfrak{b} = c_Z + \mathbf{R}(X^* - Y^*)$. Define $\mathfrak{h} = \mathbf{R}H^* + c_Z$. Then $\mathfrak{h}$ is an Iwasawa Cartan subalgebra and if $\mu = \exp[\sqrt{-1}\,\pi(X^* + Y^*)/4]$ then $\mathfrak{h}_c^\mu = \mathfrak{b}_c$. Furthermore $\tilde{\alpha}^\mu = \tilde{\alpha} \circ \mu$ is a real root of $\Delta(\mathfrak{g}_c, \mathfrak{h}_c)$ since μ leaves c_Z invariant and $\tilde{\alpha}$ vanishes on c_Z. The Cartan subalgebra $\mathfrak{h}$ is conjugate to $\mathfrak{j}$ by some element in G.

Conversely, let $\alpha \in \Delta(\mathfrak{g}_c, \mathfrak{j}_c)$ be a real root and $E_{\pm\alpha}$ be the corresponding root vectors. Since $|\rho_\alpha| = 1$ we can choose $t_\alpha \in \mathbf{R}$ such that $\exp[2\sqrt{-1}\ t_\alpha] = \rho_\alpha$. Let $h_\alpha = \exp[\sqrt{-1}\ t_\alpha H_\alpha']$. Then $\tau(h_\alpha) = h_\alpha$. If we replace the root vectors E_β by $\mathrm{Ad}\ h_\alpha(E_\beta)$ the Weyl normalization is preserved. In addition

$$\sigma(\mathrm{Ad}\ h_\alpha E_\alpha) = \sigma\left(\exp(\sqrt{-1}\ t_\alpha)E_\alpha\right) = \exp(-\sqrt{-1}\ t_\alpha)\rho_\alpha E_\alpha$$
$$= \exp\left(\sqrt{-1}\ t_\alpha\right)E_\alpha = \mathrm{Ad}\ h_\alpha E_\alpha.$$

Similarly $\sigma(\operatorname{Ad} h_\alpha E_{-\alpha}) = \operatorname{Ad} h_\alpha E_{-\alpha}$ so $E_{\pm\alpha} \in \mathfrak{g}$. Then

$$\mathfrak{b} = \mathbf{R}(E_\alpha - E_{-\alpha}) + \mathfrak{a}_K$$

is a compact Cartan subalgebra. Finally the fact that Ad ν maps $\mathfrak{b}_c$ into $\mathfrak{j}_c$ follows from an elementary calculation in sl(2, **C**). This completes the proof.

From here on α will always denote a positive real root. Fix a positive restricted root $\lambda \in \Delta(\mathfrak{g}, \mathfrak{a})$ such that the only multiples of λ that are also roots are $\pm\lambda$ and possibly $\pm\frac{1}{2}\lambda$. Let

$$\Delta_0 = \{\beta \in \Delta(\mathfrak{g}_c, \mathfrak{j}_c): \beta|_\mathfrak{a} = 0\},$$
$$P_1 = \{\beta \in \Delta(\mathfrak{g}_c, \mathfrak{j}_c): \beta|_\mathfrak{a} = \tfrac{1}{2}\lambda\},$$
$$P_2 = \{\beta \in \Delta(\mathfrak{g}_c, \mathfrak{j}_c): \beta|_\mathfrak{a} = \lambda\},$$
$$\Psi = \{\gamma \in \Delta_0: \alpha \pm \gamma \text{ are not roots}\}.$$

LEMMA 2.6. *If $P_1 \neq \varnothing$ then there is a real root $\alpha \in \Delta(\mathfrak{g}_c, \mathfrak{j}_c)$ such that $\alpha|_\mathfrak{a} = \lambda$. Also $H'_\lambda = H'_\alpha$ and* ad H'_λ *has eigenvalues* 0, ± 1 *and* ± 2.

PROOF. Let $\beta \in P_2$ and denote by α the functional on $\mathfrak{j}$ which equals λ on $\mathfrak{a}$ and 0 on $\mathfrak{a}_K$. We have to show that $\alpha \in \Delta$. Assume not. Then for any $\alpha_1 \in P_1$, the sum $\alpha_1 + \sigma\alpha_1 \notin \Delta$. Since $\alpha_1 - \sigma\alpha_1 \notin \Delta$ by Lemma 2.3, it follows that $(\alpha_1, \sigma\alpha_1) = 0$. Since $\alpha_1 + \sigma\alpha_1 = \alpha$ we also get $2(\alpha_1, \alpha_1) = (\alpha, \alpha)$. On the other hand

$$\frac{2(\beta, \alpha_1)}{(\alpha_1, \alpha_1)} + \frac{2(\beta, \sigma\alpha_1)}{(\sigma\alpha_1, \sigma\alpha_1)} = \frac{2(\beta, \alpha)}{(\alpha_1, \alpha_1)} = \frac{2(\alpha, \alpha)}{\frac{1}{2}(\alpha, \alpha)} = 4. \tag{1}$$

At least one of the summands must be larger than 2, say $2(\beta, \alpha_1)/(\alpha_1, \alpha_1) \geqslant 2$. Since β and α_1 cannot be multiples of each other, Lemma 2.1 implies that $\beta - \alpha_1$ is a root. Corollary 2.2 implies that

$$\frac{2(\beta, \alpha_1)}{(\alpha_1, \alpha_1)} \cdot \frac{2(\beta, \alpha_1)}{(\beta, \beta)} < 4$$

and since both numbers are integers, $2(\beta, \alpha_1)/(\alpha_1, \alpha_1) < 4$. Together with (1) this implies $(\beta, \sigma\alpha_1) > 0$ so $\beta - \sigma\alpha_1$ is also a root. Both $\beta - \alpha_1$ and $\beta - \sigma\alpha_1$ restrict to $\frac{1}{2}\lambda$ and they cannot be multiples of each other. Thus we can apply Lemma 2.4 to conclude that either $\beta - \alpha_1 - (\beta - \sigma\alpha_1) = \sigma\alpha_1 - \alpha_1$ or $\beta - \alpha_1 - (\sigma\beta - \alpha_1) = \beta - \sigma\beta$ is a root. This contradicts the statement of Lemma 2.3. Thus $\alpha \in \Delta$. Since $\alpha|_{\mathfrak{a}_K} = 0$ it follows that $H'_\alpha = H'_\lambda$ and $\beta(H'_\lambda) = 2$, $\alpha_1(H'_\lambda) = 1$. This completes the proof.

Let B be the Cartan subgroup corresponding to $\mathfrak{b}$. We normalize the measures on G, $\mathfrak{g}$, J, $\mathfrak{j}$, B and $\mathfrak{b}$ as in [**15a**, §8.1.2]. Furthermore we normalize the Haar measures on any compact subgroup so that it has volume one.

Let $\mathfrak{b}^*$ denote the dual to $\sqrt{-1}\ \mathfrak{b}$. Then we define a lattice L_B in $\mathfrak{b}_c^*$ by

$$L_B = \left\{ \mu \in \mathfrak{b}_c^*: \frac{2(\mu, \beta)}{(\beta, \beta)} \in \mathbf{Z} \text{ for any } \beta \in \Delta(\mathfrak{g}_c, \mathfrak{b}_c) \right\}.$$

Then the element $\rho = \frac{1}{2} \sum_{\beta > 0} \beta$ is in this lattice. For any $\mu \in L_B$ we can define a character of B by the formula

$$\xi_\mu(\exp H) = \exp \mu(H') \quad \text{all } H \in \mathfrak{b}.$$

Let

$$B' = \{ h \in B: \xi_\beta(h) \neq 1 \text{ for any } \beta \in \Delta(\mathfrak{g}_c, \mathfrak{b}_c) \}$$

and

$$\Delta_B(h) = \xi_{-\rho}(h) \prod_{\beta > 0} (1 - \xi_\beta(h)) \quad \text{for } h \in B'.$$

For any $f \in C_c^\infty(G)$ we can define a function ϕ_f^B by

$$\Phi_f^B(h) = \Delta_B(h) \int_{G/B} f(h^x)\, dx, \qquad h \in B'.$$

This function has the following properties.

(a) $\Phi_f^B \in \mathcal{S}(B')$, the Schwartz space of B',

(b) if $W(G, B)$ denotes the Weyl group of G with respect to B then $\Phi_f^B(wh) = (\det w)\Phi_f^B(h)$ for any $w \in W(G, B)$.

The function Φ_f^B plays an important role in harmonic analysis on semisimple groups. In particular the following result will be useful later.

Lemma 2.7. *Let γ be an element of B. Let $P_\gamma = \{ \beta \in \Delta(\mathfrak{g}_c, \mathfrak{b}_c)$ such that $\xi_\beta(\gamma) = 1$ and $\beta > 0\}$ and $P_{\mathfrak{g}/\gamma} = \{ \beta > 0: \beta \notin P_\gamma \}$. Define a differential operator $\bar{\omega}_\gamma$ by $\bar{\omega}_\gamma = \prod_{\beta \in P_\gamma} H_\beta$. Then*

$$\lim_{\substack{\tilde{\gamma} \to \gamma \\ \tilde{\gamma} \in B'}} \Phi_f^B(\tilde{\gamma}; \bar{\omega}_\gamma) = k_\gamma \int_{G/G_\gamma} f(\gamma^{\dot{x}})\, d\dot{x}$$

where $k_\gamma = d_\gamma \xi_{-\rho}(\gamma) \prod_{\beta \in P_{\mathfrak{g}/\gamma}} (\xi_\beta(\gamma) - 1)$ and d_γ depends only on the measure $d\dot{x}$.

Proof. See [**4d**, p. 33].

To each element $\mu \in L_B$ there is, according to [**4c**, p. 281], associated an invariant central eigendistribution Θ_μ which is a locally summable function on G and analytic on the set of regular points. If Θ_μ is regular i.e. $w\mu \neq \mu$ for any $w \in W(G_c, B_c)$ then Θ_μ is the character of a discrete series representation of G. For the explicit form of Θ_μ for real rank 1 see [**12**].

Let W_I be the Weyl group corresponding to the imaginary roots of $\mathfrak{j}$. Then W_I can be identified with the Weyl group $W(M, A_K)$ where $M = \text{Cent}_K A$. A unitary character $\chi \in \hat{A}_K$ is called regular if $w\chi \neq \chi$ for any $w \in W_I$. To each pair (χ, ν) where $\chi \in \hat{A}_K$ is regular and $\nu \in \hat{A} \approx \mathbf{R}$ there is associated a principal series character which we denote by $T^{(\chi,\nu)}$.

Let $q \in G$ be an arbitrary element. According to [3, p. 235], the group $G_q = \mathrm{Cent}_G q$ is unimodular so that the homogeneous space G/G_q carries an invariant measure. In [**11b**] it was proved that the integral

$$T_q(f) = \int_{G/G_q} f\left(q^{\dot{x}}\right) d\dot{x}$$

is convergent for any $f \in C_c(G)$. In particular, this implies that T_q is a tempered invariant distribution on G. We normalize the measures dx^* on G/G_q and dy on G_q so that

$$\int_G f(x)\, dx = \int_{G/G_q} \int_{G_q} f(x^*y)\, dx^*\, dy$$

for any $f \in C_c(G)$.

3. Conjugacy classes of nilpotent elements. An element $X \in \mathfrak{g}$ is called nilpotent if $\operatorname{ad} X$ is a nilpotent transformation of $\mathfrak{g}$. A triple $\{X, H, Y\}$ is called a Lie triple if the following relations hold:

$$[H, X] = 2X, \quad [H, Y] = 2Y \quad \text{and} \quad [X, Y] = -H.$$

Then the real algebra generated by X, H and Y is isomorphic to $\mathrm{sl}(2, \mathbf{R})$. The Jacobson-Morozov theorem [**8a**] implies that any nilpotent element X in a complex semisimple Lie algebra $\mathfrak{g}_c$ can be embedded in a Lie triple $\{X, H, Y\}$. If in addition $X \in \mathfrak{g}$ then H and Y can be chosen to be in $\mathfrak{g}$ as well.

Two Lie triples $\{X, H, Y\}$ and $\{X_1, H_1, Y_1\}$ are said to be conjugate by G or real conjugate if there is $x \in G$ such that $X^x = X_1$, $Y^x = Y_1$ and $H^x = H_1$.

The conjugacy classes of Lie triples and nilpotent elements are characterized in the next proposition.

PROPOSITION 3.1. *Two Lie triples, $\{X, H, Y\}$ and $\{X_1, H_1, Y_1\}$, are conjugate if and only if their nilpotent components X and X_1 are conjugate. The Lie triples are conjugate by G_c if and only if H and H_1 are conjugate by G_c. The Lie triples are conjugate by G if and only if $Z = X - Y$ and $Z_1 = X_1 - Y_1$ are conjugate.*

PROOF. The proof of the first two statements is contained in [**8a**]. The last statement was formulated by R. Rao and proved by B. Kostant based on [**8b**]. Since no proof seems to have appeared in print, we give one for completeness.

Assume that X_1 and X are conjugate by G. Then by the previous statement, there is $x \in G$ such that $X_1^x = X$, $Y_1^x = Y$ and $H_1^x = H$. Thus $(X_1 - Y_1)^x = X - Y$. Conversely, assume that there is $x \in G$ such that $(X_1 - Y_1)^x = X - Y$. By Theorem 6 in **[9]** we may assume that the two Lie triples are normalized so that $\theta X = -Y$ and $\theta X_1 = -Y_1$. Then $X - Y$ and $X_1 - Y_1$ are in $\mathfrak{k}$ and in addition we may assume that $x \in K$. We then replace the Lie triples X_1, H_1, Y_1 by X_1^x, H_1^x, Y_1^x and call it X_1, H_1, Y_1 once again. Thus we assume $X_1 - Y_1 = X - Y$. Since $X - Y$ is conjugate by $SL(2, \mathbf{C})$ to $\sqrt{-1}\ H$ we replace the Lie triples by the Lie triples

$$\left\{ \tfrac{1}{2}(X + Y + iH), i(X - Y), \tfrac{1}{2}(X + Y - iH) \right\}$$

and

$$\left\{ \tfrac{1}{2}(X_1 + Y_1 + iH_1), i(X_1 - Y_1), \tfrac{1}{2}(X_1 + Y_1 - iH_1) \right\}.$$

Let $\mathfrak{k}_c$ and $\mathfrak{p}_c$ be the complexifications of $\mathfrak{k}$ and $\mathfrak{p}$. Furthermore let τ_c be the complex linear map defined by $\tau_c|_{\mathfrak{k}_c} = \mathrm{id}$ and $\tau_c|_{\mathfrak{p}_c} = -\mathrm{id}$. We then note that, due to the assumptions on G_c, the map τ_c extends to a map on G_c. Let K_τ be the set of fixed points of τ_c and $\tilde{K}_\tau$ the image of K_τ by the adjoint map. If $\tilde{K}$ is the connected component of $\tilde{K}_\tau$ then by Proposition 1 in **[8b]** we can write $\tilde{K}_\tau = F \cdot \tilde{K}$ where F is the set of elements of order two in $\exp \operatorname{ad} \mathfrak{a}_c$. Also, $\tilde{K}_\tau$ is the set of fixed points of τ_c in the adjoint group of $\mathfrak{g}_c$. In the terminology of **[8b]** the two Lie triples are called normal. Due to the remark following Proposition 4 in **[8b]** the two Lie triples are conjugate by some element $\tilde{x} \in \tilde{K}_\tau$. We note that $\tilde{K} = \operatorname{Ad} K \exp\sqrt{-1}\ \operatorname{ad} \mathfrak{k}$. Since $K \subseteq G$ is connected we may assume $\tilde{x} = f \exp\sqrt{-1}\ \operatorname{ad} Z$ where $Z \in \mathfrak{k}$. Since $(X - Y)^{\tilde{x}} = X - Y$ we also have $\tau\{(X - Y)^{\tilde{x}}\} = \tau(X - Y)$ so $(X - Y)^{\tau(\tilde{x})} = X - Y$. But $\tau(\tilde{x}) = f \exp(-\sqrt{-1}\ \operatorname{ad} Z)$ so $\tilde{x}^{-1}\tau(\tilde{x}) = \exp 2\sqrt{-1}\ \operatorname{ad} Z$ also centralizes $X - Y$. But then

$$X - Y = \exp 2\sqrt{-1}\ \operatorname{ad} Z(X - Y) = \cosh(2\sqrt{-1}\ \operatorname{ad} Z)(X - Y) + \sinh(2\sqrt{-1}\ \operatorname{ad} Z)(X - Y).$$

Since the first term on the right-hand side is in $\mathfrak{k}$ and the second in $\sqrt{-1}\ \mathfrak{k}$ it follows that

$$\sinh(2\sqrt{-1}\ \operatorname{ad} Z)(X - Y) = 0.$$

But $2\sqrt{-1}\ \operatorname{ad} Z$ has real eigenvalues only, so $\operatorname{ad} Z(X - Y) = 0$. This also implies $f \cdot (X - Y) = X - Y$ so f leaves the entire Lie triple fixed. Thus we also get $\exp(\sqrt{-1}\ \operatorname{ad} Z)H = H_1$. Again

$$H_1 = \exp(\sqrt{-1}\ \operatorname{ad} Z)H = \cosh(\sqrt{-1}\ \operatorname{ad} Z)H + \sinh(\sqrt{-1}\ \operatorname{ad} Z)H$$

so $\sinh(\sqrt{-1}\ \operatorname{ad} Z)H = 0$ since it is contained in $\sqrt{-1}\ \mathfrak{p}$ (while the other term is in $\mathfrak{p}$). But this implies $(\operatorname{ad} Z)H = 0$ so $H = H_1$. Therefore the two Lie triples must coincide and the proof is complete.

We now determine the conjugacy classes of nilpotent elements in a semisimple Lie algebra of real rank one. Let λ be the restriction of the real positive root α to $\mathfrak{a}$ or in the absence of a real root let λ be the restriction of a simple positive root of $\Delta(\mathfrak{g}_c, \mathfrak{j}_c)$. Define $H'_\lambda = 2H_\lambda/(\lambda, \lambda)$ as in §2. Let $g_i = \{Z \in g\colon [H'_\lambda, Z] = iZ\}$.

THEOREM 3.2. *The real conjugacy classes of nilpotent elements are given by*
(1) $\{X, H'_\lambda, Y\}$ *where* $X \in g_2$ *and* $\{-X, H'_\lambda, -Y\}$ *when* $\dim g_2 = 1$,
(2) $\{X_1, 2H'_\lambda, Y_1\}$ *where* $X_1 \in g_1$.

PROOF. Let $\{X, H, Y\}$ be an arbitrary Lie triple. Since X, H and Y generate a semisimple algebra isomorphic to sl(2, $\mathbf{R}$), ad H must have integral eigenvalues only. Furthermore Theorem 6 in [9] implies that we can normalize the Lie triple so that $\tau X = -Y$ and $\tau H = -H$. Since $H \in \mathfrak{p}$ we can conjugate the Lie triple by an element in K so that $H = \varepsilon H'$ where $\varepsilon > 0$. We note that ad H'_λ has eigenvalues 0, ± 2 and possibly ± 1. On the other hand, ad H has an eigenvalue equal to $+2$ and thus we can only have $\varepsilon = 1$ or $\varepsilon = 2$. Furthermore, if $\varepsilon = 2$ then ad H'_λ has to have eigenvalue $+1$ so in the notation of §2 we have $P_1 \neq \varnothing$. We thus have two types of Lie triples.
(1) $\{X_1, 2H'_\lambda, Y_1\}$ where $X_1 \in \mathfrak{g}_1$,
(2) $\{X, H'_\lambda, Y\}$ where $X \in \mathfrak{g}_2$.
Due to Proposition 3.1 the two types are in different conjugacy classes. By using Theorem 8.11.3 in [14] we conclude that the Lie triples of type (1) are one conjugacy class and the Lie triples of type (2) are one conjugacy class when $\dim g_2 > 1$. It remains to show that if $\dim g_2 = 1$ then $\{X, H, Y\}$ is not conjugate to $\{-X, H, -Y\}$. Suppose that there is $x \in G$ such that $X^x = -X$, $H^x = H$ and $Y^x = -Y$. Then $x \in M = \mathrm{Cent}_K H$. Let $\mathfrak{m}$ be the Lie algebra of M. Then $[\mathfrak{m}, X] = 0$, for otherwise $\dim g_2 > 1$. This implies that M_0, the connected component of M, centralizes X. Let $G_c(H)$ be the centralizer of H in G_c and $g_c(H)$ its Lie algebra. Since $G_c(H)$ is the centralizer of a torus, it is connected. Its Lie algebra has Cartan decomposition

$$g_c(H) = (\mathfrak{m} + \sqrt{-1}\ \mathbf{R}H) + (\sqrt{-1}\ \mathfrak{m} + \mathbf{R}H).$$

Thus

$$G_c(H) = M \exp\{\sqrt{-1}\ \mathbf{R}H\}\exp\{\sqrt{-1}\ \mathfrak{m} + \mathbf{R}H\}.$$

Thus $M = M_0 \exp\{\sqrt{-1}\ \mathbf{R}H\} \cap K$. Suppose $h = \exp\{\sqrt{-1}\ tH\} \in K$ for some $t \in \mathbf{R}$. Then $\sigma(\exp\{\sqrt{-1}\ tH\}) = \exp\{\sqrt{-1}\ tH\}$ so $\exp\{\sqrt{-1}\ tH\} = \exp\{-\sqrt{-1}\ tH\}$ or $\exp\{2\sqrt{-1}\ tH\} = e$. Since H as an element of g_c is contained in a subalgebra isomorphic to sl(2, $\mathbf{C}$) it follows that $t = k\pi$ for some $k \in \mathbf{Z}$. But then

$$X^h = \exp\{2\sqrt{-1}\ t\}X = \exp\{2\sqrt{-1}\ k\pi\}X = X.$$

Thus there cannot be such $x \in M$ and the proof is complete.

EXAMPLE. Let $\mathfrak{g} = \mathrm{su}(2, 1)$. Then representatives of all conjugacy classes are given by

$$X = \begin{bmatrix} 0, & 0, & 0 \\ 0, & i/2, & i/2 \\ 0, & -i/2, & -i/2 \end{bmatrix}, \quad H'_\lambda = \begin{bmatrix} 0, & 0, & 0 \\ 0, & 0, & -1 \\ 0, & -1, & 0 \end{bmatrix},$$

$$Y = \begin{bmatrix} 0, & 0, & 0 \\ 0, & i/2, & -i/2 \\ 0, & i/2, & -i/2 \end{bmatrix}. \tag{1}$$

In this case X and $-X$ are not conjugate.

$$X = \begin{bmatrix} 0, & 1, & 1 \\ -1, & 0, & 0 \\ 1, & 0, & 0 \end{bmatrix}, \quad 2H'_\lambda = \begin{bmatrix} 0, & 0, & 0 \\ 0, & 0, & -2 \\ 0, & -2, & 0 \end{bmatrix},$$

$$Y = \begin{bmatrix} 0, & 1, & -1 \\ -1, & 0, & 0 \\ -1, & 0, & 0 \end{bmatrix}. \tag{2}$$

REMARK 3.3. In the case when $P_1 \neq \varnothing$ let $\alpha_1 \in P_1$. Then the roots α_1, $\sigma\alpha_1$ and α generate a subalgebra isomorphic to su(2, 1). Thus we can select representatives for the conjugacy classes of nilpotents to be as in the example above. If $P_1 = \varnothing$ then the representatives can be chosen in $\mathrm{sl}(2, \mathbf{R}) \subseteq \mathrm{su}(2, 1)$ to have the same expression as X, H and Y in part (1) of the example.

4. The orbit of the semisimple element Z. With the notation as before let $\{X, H, Y\}$ be a Lie triple normalized as before and let $Z_0 = X - Y$. Let G_Z be the centralizer of Z_0 in G and $\mathfrak{g}_Z$ its Lie algebra. Since Z_0 is contained in $\mathfrak{k}$, G_Z is connected and has a Cartan decomposition $G_Z = K_Z S_Z$. Let $\mathfrak{g}_Z = \mathfrak{k}_Z + \mathfrak{p}_Z$ be the corresponding decomposition for the Lie algebra. Let $L = \mathrm{Cent}_G\{X, H, Y\}$ and $\mathfrak{L}$ be its Lie algebra. If $g_i = \{Z\colon \mathrm{ad}\, HZ = iZ\}$ then let $g_{\mathrm{ev}} = \sum_{i \text{ even}} g_i$ and $g_{\mathrm{o}} = \sum_{i \text{ odd}} g_i$. Then g_{ev} is a τ-invariant subalgebra and therefore reductive. Let $g_{\mathrm{ev}} = \mathfrak{k}_{\mathrm{ev}} + \mathfrak{p}_{\mathrm{ev}}$ be the Cartan decomposition and $\mathfrak{p}_{\mathrm{o}}$ be the orthogonal complement of $\mathfrak{p}_{\mathrm{ev}}$ in $\mathfrak{p}$ with respect to the Cartan-Killing form.

LEMMA 4.1. *Let* (π, V) *be an irreducible finite dimensional representation of* $\mathrm{sl}(2, \mathbf{C})$ *of highest weight n. Let* $V^{\mathrm{o}} = \mathrm{Cent}\, Z_0$ *and* $V_i = \{v\colon \pi(H)v = iv\}$. *Then* $V^{\mathrm{o}} = \{0\}$ *if n is odd and* V^{o} *is isomorphic to* V_n *when n is even.*

PROOF. Let $v = v_n + v_{n-2} + \cdots + v_{-n}$ be an element of V^{o}. From representation theory of sl(2, C) e.g. [**13**, Lemma A and Theorem B, p. 28] we know that $\dim V_i = 1$ for $-n \leqslant i \leqslant n$, $i \equiv n \pmod 2$, and $V_i = \{0\}$ otherwise. Furthermore, $\pi(X)V_n = \pi(Y)V_{-n} = \{0\}$ and $\pi(X)$, $\pi(Y)$ are isomorphisms from V_i to V_{i+2} and V_i to V_{i-2} respectively. The equation $\pi(Z_0)v = 0$ decomposes according to eigenspaces of $\pi(H)$ in the following way. Let i be an eigenvalue of $\pi(H)$. Then the component of $\pi(Z_0)v$ in V_i gives the equation

$$\pi(X)v_{i-2} - \pi(Y)v_{i+2} = 0. \qquad (*)$$

Let j be the largest integer so that $v_j \neq 0$. For V_{j+2} we get the equation $\pi(X)v_j = 0$. Thus $j = n$ if $v \neq 0$. Furthermore, for V_n we also get $\pi(X)v_{n-2} = 0$ so $v_{n-2} = 0$. Equation (*) implies $v_k = 0$ if $n - k \neq 0 \pmod 4$. Let k be the integer such that $v_{n-4k} \neq 0$ but $v_{n-4k-4} = 0$. Then $\pi(X)v_{n-4k-4} - \pi(Y)v_{n-4k} = 0$ so $\pi(Y)v_{n-4k} = 0$. Thus $n - 4k = -n$ or $n = 2k$. This proves the lemma.

COROLLARY 4.2. *The linear subspace* $\mathbf{R}H + [H, \mathfrak{k}_Z]$ *is orthogonal to* $\mathfrak{p}_Z$. *Let* $\mathfrak{p}_1$ *be the orthogonal complement of* $\mathbf{R}H + [H, \mathfrak{k}_Z] + \mathfrak{p}_Z$ *in* $\mathfrak{p}$ *and* $W \in \mathfrak{p}_1$, $W \neq 0$. *Then* $(\mathbf{R}W)^L = \mathfrak{p}_1$.

PROOF. Let $Z_1 \in \mathfrak{k}_Z$ and $Z_2 \in \mathfrak{p}_Z$. Then $B_\tau([H, Z_1], Z_2) = B_\tau(H, [Z_1, Z_2])$. Then $[Z_1, Z_2] \in \mathfrak{g}_Z$. Let $Z \in \mathfrak{g}_Z$ be arbitrary. Then

$$B_\tau(H, Z) = B([X, Y], Z) = B(Y, [X, Z]) = B(Y, [Y, Z]) = 0$$

since $[Z_0, Z] = 0$ which is the same as $[X, Z] = [Y, Z]$. Thus $[H, \mathfrak{k}_Z]$ is orthogonal to $\mathfrak{p}_Z$ and also H is orthogonal to $\mathfrak{g}_Z$. Thus the first statement in the corollary is proved. For the second we note that since $L \subseteq K$ and $\mathbf{R}H + [H, \mathfrak{k}_Z] + \mathfrak{p}_Z$ is L-invariant, $\mathfrak{p}_1$ is also L-invariant. Next we show that

(1) $$[H, \mathfrak{k}_Z] + \mathfrak{p}_Z = \{Z - \tau Z : Z \in (\ker \operatorname{ad} X)^{\mathrm{ev}}\}.$$

We recall that $\mathfrak{g}$ is an algebra of real rank 1 and therefore $\mathfrak{g}_Z$ is generated by elements of the form $v_4 + v_0 + v_{-4}$ and $v_2 + v_{-2}$ where v_2 and v_4 centralize ad X. If $v_2 + v_{-2} \in \mathfrak{k}_Z$ or $v_4 + v_0 + v_{-4} \in \mathfrak{k}_Z$ then $[H, v_2 + v_{-2}] = 2v_2 - 2v_{-2}$ and $[H, v_4 + v_0 + v_{-4}] = 4v_4 - 4v_{-4}$ are in $\mathfrak{p}$ and have the desired form. The statement follows from these two relations. This also implies that

(2) $\mathfrak{p}_1 = \{Z - \tau Z\colon\ Z \in g^+$ and perpendicular to $(\ker \operatorname{ad} X)^{\mathrm{ev}}\}$ where $g^+ = \sum_{i<0} g_i$.

Let $X_1 \in g^+$ be such that $W = X_1 - \tau X_1 \in \mathfrak{p}_1$. Then Theorem 8.11.3 in **[14]** has as a simple consequence the fact that the group L acts transitively on the unit sphere of the orthogonal of $(\ker \operatorname{ad} X)^{\mathrm{ev}}$. Then in view of the description of $\mathfrak{p}_1$ given in (2) we get $(\mathbf{R}W)^L = p_1$.

PROPOSITION 4.3. *Let* $\mathfrak{p}_Z^\perp$ *be the orthogonal complement of* $\mathfrak{p}_Z$ *in* $\mathfrak{p}$. *Then the map* $\psi\colon K_Z \times \mathbf{R}^2 \to \mathfrak{p}_Z^\perp$ *given by* $\psi(m, s_1, s_2) = (s_1 W + s_2 H)^m$ *is onto* $\mathfrak{p}_Z^\perp$.

PROOF. We calculate the differential of ϕ. It equals

$$D\psi_{(m,s_1,s_2)}(Z, t_1, t_2) = \operatorname{Ad} m\big(s_1[Z, W] + s_2[Z, H] + t_1W + t_2H\big).$$

In view of Corollary 4.2 this map is a submersion at any point (m, s_1, s_2) where $s_1s_2 \neq 0$. Let $\mathfrak{D}$ be the set of points at which ψ is a submersion. Then $\mathfrak{D}$ is dense in $K_Z \times \mathbf{R}^2$ and $\psi(\mathfrak{D})$ is open in $\mathfrak{p}_Z^\perp$. Since K_Z is a compact group, $\psi(K_Z \times \mathbf{R}^2)$ is also closed. We now examine the boundary of $\psi(\mathfrak{D})$. It is contained in $\mathbf{R}W^{K_Z} \cup \mathbf{R}H^{K_Z}$ which is a union of two submanifolds namely $\mathbf{R}^xW^{K_Z}$ and $\mathbf{R}^xH^{K_Z}$, and $\{0\}$. If we can show that the two submanifolds have at least codimension 2 in $\mathfrak{p}_Z^\perp$ then it follows that ψ is onto. Indeed, let $q \in \mathfrak{p}_Z^\perp \setminus \psi(K_Z \times \mathbf{R}^2)$. Then we can join q to any element of the interior of $\psi(K_Z \times \mathbf{R}^2)$ by a curve that does not intersect $\mathbf{R}W^{K_Z} \cup \mathbf{R}H^{K_Z}$ (see Proposition 9.4 in [5]). This is a contradiction so $\mathfrak{p}_Z^\perp = \psi(K_Z \times \mathbf{R}^2)$.

It remains to show that the boundary has codimension at least 2. By Corollary 4.2 we have the direct sum decomposition

(1) $\mathbf{R}H + [H, \mathfrak{k}_Z] + \mathfrak{p}_Z + \mathfrak{p}_1 = \mathfrak{p}$.

Consider the case where $\dim \mathfrak{p}_1 > 1$ first. Then $\mathbf{R}H + [H, \mathfrak{k}_Z]$ has codimension equal to $\dim \mathfrak{p}_1$ so $(\mathbf{R}^xH)^{K_Z}$ has codimension at least 2.

We have the following relations.

(2) $B_\tau([W, \mathfrak{k}_Z], H) = 0$,

(3) $B_\tau([W, \mathfrak{k}_Z], \mathfrak{p}_Z) = 0$,

(4) $B_\tau([W, \mathfrak{k}_Z], W) = 0$.

From (2) we obtain that H is orthogonal to $\mathbf{R}W + [W, \mathfrak{k}_Z]$. From relation (3) and $W \in \mathfrak{p}_1$ we get

(5) $\mathbf{R}W + [W, \mathfrak{k}_Z] \subseteq [H, \mathfrak{k}_Z] + \mathfrak{p}_1$.

In view of $\mathbf{R}W + [W, \mathfrak{L}] = \mathfrak{p}_1$ let $[W, \mathfrak{k}_Z]^\perp$ be the perpendicular complement of $\mathfrak{p}_1$ in $\mathbf{R}W + [W, \mathfrak{k}_Z]$. Then $[W, \mathfrak{k}_Z]^\perp \subseteq [H, \mathfrak{k}_Z]$. If $[W, \mathfrak{k}_Z]^\perp \neq [H, \mathfrak{k}_Z]$ then we can find an element orthogonal to $\mathbf{R}W + [W, \mathfrak{k}_Z]$ and H so the codimension of $(\mathbf{R}^xW)^{K_Z}$ is at least 2.

If on the other hand $[W, \mathfrak{k}_Z]^\perp = [H, \mathfrak{k}_Z]$ we have equality in (5). Then $\mathbf{R}W + s_1[W, \mathfrak{k}_Z] = [H, \mathfrak{k}_Z] + \mathfrak{p}_1$ for $s_1 \neq 0$ so $(m, s_1, s_2) \in \mathfrak{D}$ for any $s_1 \neq 0$. Evidently $\{0\}$ has codimension at least 2.

Next we consider the case $\dim \mathfrak{p}_1 = 1$. Then $W^L = W$ and $\mathbf{R}H + [H, k_Z]$ is the orthogonal of $\mathbf{R}W + \mathfrak{p}_Z$ in $\mathfrak{p}$ so if $s_2 \neq 0$ then $(m, s_1, s_2) \in \mathfrak{D}$. Again $\{0\}$ has codimension at least 2. The proof is now complete.

We now recall the following result.

PROPOSITION 4.4. *Let G be a connected semisimple Lie group and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be its Cartan decomposition. Let $\mathfrak{p}'$ be a linear subspace of p such that $[X, [X, Y]] \in \mathfrak{p}'$ for any X, $Y \in \mathfrak{p}'$. Let $\mathfrak{p}^\perp$ be the orthogonal of $\mathfrak{p}'$ in $\mathfrak{p}$. Then G decomposes topologically into $G = K \cdot F \cdot E$ where $F = \exp \mathfrak{p}^\perp$ and $E = \exp \mathfrak{p}'$.*

PROOF. See [9, Theorem 3, p. 40].

By applying this result to the case considered here we get the following corollaries.

COROLLARY 4.5. *Let* $F = \exp \mathfrak{p}_Z^\perp$ *and* $S_Z = \exp \mathfrak{p}_Z$. *Then*

$$G = KFS_Z = K \exp(\mathbf{R}W + \mathbf{R}H)G_Z.$$

PROOF. We note that $\mathfrak{p}_Z$ is the $\mathfrak{p}$-component of the subalgebra $\mathfrak{g}_Z$ so it satisfies the properties of $\mathfrak{p}'$ in Proposition 4.4. Thus we can write $G = KFS_Z$. By Proposition 4.3 $\mathfrak{p}_Z^\perp = (\mathbf{R}H + \mathbf{R}W)^{K_Z}$ so

$$\exp \mathfrak{p}_Z^\perp \subseteq K_Z \exp(\mathbf{R}H + \mathbf{R}W)K_Z.$$

Therefore, $G = KK_Z \exp(\mathbf{R}H + \mathbf{R}W)K_Z S_Z = K \exp(\mathbf{R}H + \mathbf{R}W)G_Z$.

COROLLARY 4.6. *Let* $\mathfrak{g} = \mathrm{su}(2, 1)$,

$$H = \begin{bmatrix} 0&0&0 \\ 0&0&1 \\ 0&1&0 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 0&0&1 \\ 0&0&0 \\ 1&0&0 \end{bmatrix}.$$

Let $A = \exp \mathbf{R}H$ *and* $F = \exp \mathbf{R}W$. *Then the maps* ϕ_1, ϕ_2: $A \times F \to \exp(\mathbf{R}H + \mathbf{R}W)$ *given by* $\phi_1(e, f) = fef$ *and* $\phi_2(e, f) = efe$ *are isomorphisms.*

PROOF. This corollary can be regarded as a particular case of Proposition 4.4. We give a sketch of the proof. Theorem 1 on p. 37 in [9] shows that ϕ maps $A \times F$ into $\exp(\mathbf{R}H + \mathbf{R}W)$. Let $p(t)$ be a path in $\exp(\mathbf{R}H + \mathbf{R}W)$ joining the origin to some arbitrary element x. We denote by $|x|$ the Riemannian distance between x and the origin under the metric given by $(ds/dt)^2 = \mathrm{Tr}(p^{-1}\dot{p})$. Then the proof of Theorem 3 in [9] shows that $|efe| > \sup(|e|, |f|)$. If $e_n f_n e_n$ converges to x, $|e_n f_n e_n|$ is bounded so $|e_n|$ and $|f_n|$ are also bounded so e_n has a convergent subsequence and so does f_n. Let $e_n \to e \in A$ and $f_n \to f \in F$. Then $efe = x$. This shows that the image of ϕ is closed. A calculation of the Jacobian of ϕ shows that the image is also open. The rest of the proof is as in Theorem 1 in [9].

We now put all these results together in the following theorem.

THEOREM 4.7. *Let G be a real semisimple group of rank one. With notation as before,* $G = KA^+ \exp \mathbf{R}WG_Z$. *Let* L_0 *be the normalizer of* $\mathbf{R}W$ *in L. Then the map* ψ: $K/L_0 \times \mathbf{R}^+ \times \mathbf{R}^\times \times G_Z \to G$ *defined by* $\psi(k, s_1, s_2, m) = k \exp s_2 H \exp s_1 Wm$ *has finite fiber of constant cardinality.*

PROOF. Let $x \in G$ be such that $x = ks$ where $k \in K$, $s \in S$ and $S = \exp \mathfrak{p}$. Let $x = k_1 zm$ be the decomposition in Corollary 4.5. By the remark at the end of §3 we may assume that the Lie triple $\{X, H, Y\}$ and W are contained in a τ-invariant subalgebra isomorphic to su(2, 1). Let K' be the maximal compact subgroup of SU(2, 1) and G'_{Z_0} be the centralizer of Z_0 in

SU(2, 1). In this case, G'_{Z_0} is compact. We can then write

$$G = (K/K')K' \exp(\mathbf{R}H + \mathbf{R}W)G'_{Z_0}(G'_{Z_0} \setminus G_{Z_0})$$
$$= (K/K')\mathrm{SU}(2, 1)(G'_{Z_0} \setminus G_{Z_0}).$$

In view of this decomposition it is enough to show the result for SU(2, 1). Assume that $x \in \mathrm{SU}(2, 1)$. Then $\tau(x) = ks^{-1}$ and $\tau(x)^{-1}x = s^2$. Since in this case $G_{Z_0} = K_{Z_0}$, $\mathfrak{p}_Z^\perp = \mathfrak{p}$. Therefore, by Proposition 4.3, s^2 can be conjugated into $\exp(\mathbf{R}H + \mathbf{R}W)$. Let $m \in G_Z$ and $Z \in \exp(\mathbf{R}H + \mathbf{R}W)$ be such that $s = m^{-1}zm$. By Corollary 4.6 we can write $z^2 = fe^2f$ with $e \in A$ and $f \in \exp \mathbf{R}W$. Then $xm^{-1}f^{-1}e^{-1} \in K$ since

$$\tau(xm^{-1}f^{-1}e^{-1}) = \tau(x)m^{-1}fe = ks^{-1}m^{-1}fe = km^{-1}z^{-1}fe$$
$$= km^{-1}z^{-1}z^2f^{-1}e^{-1} = ksm^{-1}f^{-1}e^{-1} = xm^{-1}f^{-1}e^{-1}.$$

Let $k' = xm^{-1}f^{-1}e^{-1}$. Then $x = k'efm$ so $G = KA \exp \mathbf{R}WG_Z$.

Suppose that $x = k_1e_1f_1m_1 = k_2e_2f_2m_2$. Then $\tau(x)^{-1}x = m_1^{-1}f_1e_1^2f_1m_1 = m_2^{-1}f_2e_2^2f_2m_2$. By Corollary 4.6, $f_1e_1^2f_1$ and $f_2e_2^2f_2$ are elements of $\exp(\mathbf{R}H + \mathbf{R}W)$ so we have to investigate the relationship $(t_1W + t_2H)^m = s_1W + s_2H$ where t_1 and t_2 are nonzero. Due to the proof of Proposition 4.3, $(t_1W + t_2H)^{K_Z}$ is transverse to $\mathbf{R}H + \mathbf{R}W$ (t_1 and t_2 are fixed) since $[W, \mathfrak{k}_Z]$ and $[H, \mathfrak{k}_Z]$ are both orthogonal to $\mathbf{R}H + \mathbf{R}W$.

This also implies that the intersection has dimension 0. Since K_Z is compact, $(t_1W + t_2H)^{K_Z} \cap (\mathbf{R}H + \mathbf{R}W)$ is a finite set. We have to show that it has constant cardinality and also compute the dimension of the centralizer of $t_1W + t_2H$ in K_Z. Let $\mathfrak{L}_0$ be the Lie algebra of L_0. There are two cases.

Case I. ad H has odd eigenvalues. In this case K_Z leaves $\mathfrak{p}_1$ invariant since $\mathfrak{k}_Z \subseteq g^{\mathrm{ev}}$ while $\mathfrak{p}_1$ is generated by elements of the form $X_1 - \tau X_1$ where $X_1 \in \mathfrak{g}^{\mathrm{ev}}$. Let $m \in K_Z$. If $(t_1W + t_2H)^m = s_1W + s_2H$ then $t_1W^m = s_1W$ and $t_2H^m = s_2H$. This implies that the connected component of the centralizer of $t_1W + t_2H$ is contained in L_0. Since L acts transitively on the unit sphere in g we can find $l \in L$ such that $W^l = -W$. Then $(t_1W + t_2H)^l = (-t_1W + t_2H)$. Also

$$\operatorname{card}\{(t_1W + t_2H)^{K_Z} \cap (\mathbf{R}W + \mathbf{R}H)\}$$
$$= \operatorname{card}\{(-t_1W - t_2H)^{K_Z} \cap (\mathbf{R}W + \mathbf{R}H)\}.$$

Since the map $\phi\colon K_Z \times (\mathbf{R}^x)^2 \to \mathfrak{p}_Z^\perp$ is a submersion, ϕ has constant fibers on each connected component of $(\mathbf{R}^x)^2$.

Case II. ad H has even eigenvalues only. We calculate the dimension of $K/L_0 \times A \times \exp \mathbf{R}W \times G_Z$. We already know that the map ϕ is a submersion. We then have the following relations

(1) $\dim \mathfrak{k} = \dim g_0 - 1 + \dim g_2 + \dim g_4$,

(2) $\dim \mathfrak{g}_Z = \dim g_0$,

(3) $\dim \mathfrak{L} - \dim \mathfrak{L}_0 = \dim \mathfrak{p}_1 - 1$,
(4) $\dim \mathfrak{p}_1 + \dim \mathfrak{p}_Z + \dim[H, \mathfrak{k}_Z] + 1 = \dim \mathfrak{p} = \dim \mathfrak{g}_2 + \dim \mathfrak{g}_4 + 1$,
(5) $\dim[H, \mathfrak{k}_Z] = \dim \mathfrak{k}_Z - \dim \mathfrak{L}$.

Then relations (3), (4) and (5) give

$$\dim \mathfrak{L}_0 = \dim \mathfrak{L} - \dim \mathfrak{p}_1 + 1 = \dim \mathfrak{g}_0 - \dim \mathfrak{g}_2 - \dim \mathfrak{g}_4 + 1$$

so

$$\dim \mathfrak{k} - \dim \mathfrak{L}_0 + 2 + \dim \mathfrak{g}_Z = \dim \mathfrak{g}_0 + 2 \dim \mathfrak{g}_2 + 2 \dim \mathfrak{g}_4 = n.$$

To show that the fiber has constant cardinality we make use of the fact that the map ϕ mentioned in Case I is a submersion so its fiber has constant cardinality on connected components of $(\mathbf{R}^\times)^2$, and the relation

$$\operatorname{card}\left((-t_1 W - t_2 H)^{L_0}\right) = \operatorname{card}\left((t_1 W + t_2 H)^{L_0}\right).$$

Let $X_2 = [X_1, X] \in \mathfrak{g}_4$ where $X_1 - \tau X_1 = W$. Then $X_2 \neq 0$ and there is l centralizing X_2 and H (by Theorem 8.11.3 in **[14]**) such that $X^l = -X$. But then $W^l = -W$. Let $(t_1 W + t_2 H)^m = s_1 W + s_2 H$ where $m \in L_0$. Then $l^{-1} m l \in L_0$ and

$$(-t_1 W + t_2 H)^{l^{-1} m l} = (t_1 W + t_2 H)^{ml} = (s_1 W + s_2 H)^l = -s_1 W + s_2 H.$$

Thus the fiber has constant cardinality. The proof of the proposition is complete once we note that in su(2, 1) the element

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{bmatrix} \in L_0$$

maps H into $-H$.

We now derive an integration formula corresponding to the decomposition in Theorem 4.7.

PROPOSITION 4.8. *Let ψ: $K/L_0 \times A^+ \times \exp \mathbf{R} W \times G_Z \to G$ be the map given by $\psi(k, a, w, m) = kawm$. Let $g(t_1, t_2) = |\det D\psi_{k,a,w,m}|$ where $k \in K/L_0$, $a = \exp t_2 H$, $w = \exp t_1 W$ and $m \in G_Z$. Let c_{Z_0} be the cardinality of the fiber of ψ at any point where $t_1, t_2 \neq 0$. Then*

$$\int_{G/G_{Z_0}} f(Z_0^x)\, dx = \frac{1}{c_{Z_0}} \int_{K/L_0} \int_{A^+} \int_{\mathbf{R}} g(t_1, t_2) f(Z_0^{kaw})\, dt_1\, da\, dk.$$

Furthermore $g(t_1, t_2) = |\det \phi|$ where ϕ is the linear map

$$\phi(Z_1, s_2, s_1, Z_2) = \operatorname{Ad} a^{-1} Z_1 + s_2 H + s_1 W + \operatorname{Ad} w^{-1} Z_2$$

for $Z_1 \in \mathfrak{k}/\mathfrak{L}_0$, $Z_2 \in \mathfrak{g}_Z$.

PROOF. In order to establish the integral formula it is enough to calculate the pullback under ψ of the Haar measure dx of G.

By Theorem 4.7, $K/L_0 \times A^+ \times \exp \mathbf{R}^{\times} W \times G_Z$ is a covering space of an open dense subspace of G and the fiber has constant cardinality. Then the following formula holds for any $f \in C_c^\infty(G)$.

$$\int_G f(x)\,dx = \frac{1}{c_{Z_0}} \int_{K/L_0} \int_{A^+} \int_{\mathbf{R}} \int_{G_Z} f(kawm) J_{k,a,w,m}\,dk\,da\,dw\,dm$$

where J is the Jacobian. Due to the fact that the measures involved are all left and right invariant we can express $J_{k,a,w,m}$ as a function of a and w alone. We get

$$D\psi_{k,a,w,m}(Z_1, s_2, s_1, Z_2) = L^*_{kawm}\big(\mathrm{Ad}(m^{-1}w^{-1}a^{-1})Z_1 + s_2\,\mathrm{Ad}(m^{-1}w^{-1})H + s_1 \mathrm{Ad}(m^{-1})W + Z_2\big)$$

where L_g^* is the differential of left multiplication by g. By using the fact that dx is left invariant and that $|\det \mathrm{Ad}\, x| = 1$ for any $x \in G$ and $|\det_{\mathfrak{g}_Z} \mathrm{Ad}(m)| = 1$ for any $m \in G_Z$ we get $|J_{k,a,w,m}| = |\det(\mathrm{Ad}\, a^{-1}Z_1 + s_2 H + s_1 W + \mathrm{Ad}\, wZ_2)|$. This is exactly the expression for $|\det \phi|$ in the statement of the proposition. The integral formula now follows from the normalization of the measures so that $dx = d_{G/G_Z} x^* \cdot d_{G_z} \dot{x}$.

5. The orbit of the nilpotent element X. We now find a decomposition and an integral formula similar to Theorem 4.7 and Proposition 4.8 for a nilpotent element X. Let $\{X, H, Y\}$ be a Lie triple as before.

LEMMA 5.1. *Let* $N_0 = \mathrm{Cent}_N X$. *If* G_X *is the centralizer of* X *in* G *then* $G_X = LN_0$. *Furthermore, if* $y \in G_X$ *is semisimple then* y *can be conjugated into* L *by some element in* N_0.

PROOF. From representation theory of sl(2, $\mathbf{C}$) it follows that $\mathrm{Cent}_{\mathfrak{g}} X = \mathfrak{L} + \mathfrak{n}_0$ where $\mathfrak{n}_0$ is the Lie algebra of N_0. Let $y \in G$ be such that $X^y = X$. Then $H^y - H \in \mathrm{Cent}_{\mathfrak{g}} X$. Therefore, $H^y = H + m_0 + n_0$ where $m_0 \in \mathfrak{L}$ is semisimple and $n \in \mathfrak{n}_0$. For any invariant polynomial p we have the relation $p(H) = p(H^y) = p(H + m_0 + n_0) = p(H + m_0)$. But ad m_0 has imaginary eigenvalues and $[H, m_0] = 0$. It follows that $m_0 = 0$. By a well-known argument, $H + n_0 = H^{n_1}$ for some $n_1 \in N_0$. Thus $H^y = H^{n_1}$ so $yn_1^{-1} \in \mathrm{Cent}_G H \cap \mathrm{Cent}_G X = L$. This establishes the decomposition $G_X = LN_0$.

Let now $y \in G_X$ be semisimple. Ad y normalizes N_0 so

$$\mathfrak{n}_0 = \mathrm{Im}(-\mathrm{Ad}\, y + 1) \oplus \ker(-\mathrm{Ad}\, y + 1).$$

Then $H^y = H + Z_0 + Z_1$ where $Z_0 = (-\mathrm{Ad}\, y + 1)Z_2$ and $(-\mathrm{Ad}\, y + 1)Z_1 = 0$. Then

$$(H + Z_2)^y = H + Z_0 + Z_1 + Z_2^y = H + Z_2 + Z_1$$

so that

$$(\operatorname{Ad} y - 1)^2(H + Z_2) = (\operatorname{Ad} y - 1)Z_1 = 0.$$

Since $\operatorname{Ad} y - 1$ is semisimple $(\operatorname{Ad} y - 1)(H + Z_2) = 0$. As $H + Z_2 = H^n$ for some $n \in N_0$ it follows that $H^n = H + Z_2 = (H + Z_2)^y = H^{ny}$ or $nyn^{-1} \in \operatorname{Cent}_G H \cap \operatorname{Cent}_G X = L$.

This completes the proof.

Lemma 5.2. *Let* $\mathfrak{n}_0$ *be the Lie algebra of* N_0 *and* $\mathfrak{n}_1$ *its orthogonal complement in* $\mathfrak{n}$. *If* $N_1 = \exp \mathfrak{n}_1$ *then* $N = N_1N_0$ *and* $G = K/LAN_1G_X$.

Proof. Let $\mathfrak{n}^{(k)} = \Sigma_{i \geqslant k}\, g_i$. Then $\mathfrak{n}^{(k)}$ is an ideal in $\mathfrak{n}$, $\mathfrak{n}^{(k)} \supseteq \mathfrak{n}^{(k+1)}$ and $\mathfrak{n}^{(1)} = n$. Furthermore, since $\operatorname{ad} H\mathfrak{n}_0 \subseteq \mathfrak{n}_0$ and $\operatorname{ad} H\mathfrak{n}_1 \subseteq \mathfrak{n}_1$ we can decompose $\mathfrak{n}^{(k)}$ into $\mathfrak{n}^{(k)} = \mathfrak{n}^{(k)} \cap \mathfrak{n}_0 + \mathfrak{n}^{(k)} \cap \mathfrak{n}_1$. Thus Lemma 1 on p. 736 in **[4a]** applies so $N = N_1N_0$. To show that this decomposition is unique it is enough to show that, if $\exp X_2 = \exp(-X_1)\exp Y_1$ where $X_2 \in \mathfrak{n}_0$ and $X_1, Y_1 \in \mathfrak{n}_1$, then $X_2 = 0$. The Campbell-Baker-Hausdorff formula states that

$$\log\left[\exp(-X_1)\exp Y_1\right] = -X_1 + Y_1 + \tfrac{1}{2}\left[-X_1, Y_1\right] + \tfrac{1}{12}\left[-X_1, \left[-X_1, Y_1\right]\right] + \cdots .$$

Suppose that $X_1 = Z + X_1'$ where $X_1' = \mathfrak{n}^{(s+1)}$ and $Y_1 = Z + Y_1'$ where $Y_1' \in \mathfrak{n}^{(s+1)}$. Then $[X_1, Y_1] = [Z, Y_1'] + [X_1', Z] + [X_1', Y_1'] \in \mathfrak{n}^{(s+2)}$. Therefore

$$\log\left[\exp(-X_1)\exp Y_1\right] \equiv Y_1' - X_1' \pmod{\mathfrak{n}^{(2+s)}}.$$

On the other hand $\log[\exp(-X_1)\exp Y_1] \equiv X_2 \pmod{\mathfrak{n}^{(s+2)}}$ which is possible only if $X_1' \equiv Y_1' \mod \mathfrak{n}^{(s+2)}$ and $X_2 \equiv 0 \mod \mathfrak{n}^{(s+2)}$. A simple induction argument then shows that $X_2 = 0$. The Iwasawa decomposition transforms into

$$G = KAN = KAN_1N_0 = K/L\ LAN_1N_0 = K/L\ AN_1LN_0 = K/L\ AN_1G_X$$

and the proof is complete.

Remark. The previous two lemmas are true for G of arbitrary rank with minor modifications.

In accordance with §4 let $W = X_1 - \tau X_1$ where $X_1 \in \mathfrak{n}$.

Proposition 5.3. *Let G be a semisimple Lie group of real rank one. Then* $G = KA\exp \mathbf{R}X_1G_X$ *and the decomposition is unique up to* $L_0 = \operatorname{Norm}_L\{\mathbf{R}X_1\} = \operatorname{Norm}_L\{\mathbf{R}W\}$.

Proof. By the proof of Corollary 4.2 we know that $\mathfrak{n}_1 = (\mathbf{R}X_1)^L$ so $N_1 = (\exp \mathbf{R}W_1)^L$. We substitute this into the decomposition of Lemma 5.2.

$$G = K/LA(\exp \mathbf{R}X_1)^LG_X = KA\exp \mathbf{R}X_1G_X.$$

Suppose $k_1a_1\exp t_1X_1l_1n_1 = k_2a_2\exp t_2X_1l_2n_2$. Then

$$k_1l_1a_1 \exp t_1X_1^{l_1^{-1}}n_1 = k_2l_2a_2 \exp t_2X_1^{l_2^{-1}}n_2.$$

Since this decomposition is unique, $a_1 = a_2$, $k_1l_1 = k_2l_2$, $\exp t_1X_1^{l_1^{-1}} = \exp t_2X_1^{l_1^{-1}}$ or $(t_1X_1)^{l_1^{-1}l_2} = t_2X_1$ and $n_1 = n_2$. Thus $l_1l_2^{-1} \in \mathrm{Norm}_L(\mathbf{R}W_1)$ so $k_2^{-1}k_1 \in \mathrm{Norm}_L(\mathbf{R}X_1)$. Thus the decomposition $G = K/L_0A \exp t_1X_1G_X$ is unique.

PROPOSITION 5.4. *Let* $\rho(H) = \frac{1}{2}\operatorname{tr} \operatorname{ad} H|_{\mathfrak{n}}$. *Then*

$$\int_{G/G_X} f(X^x)\,dx = \int_{K/L_0}\int_A e^{2\rho(\log a)}\int_{\mathbf{R}} t^{\dim \mathfrak{n}_1 - 1} f(X^{ka \exp t_1X_1})\,dt_1\,da\,dk$$

for any $f \in C_c^\infty(\mathfrak{g})$.

PROOF. Let ψ: $K/L_0 \times A \times \exp t_1W \times G_X \to G$ be the map given by $\psi(k, a, w, m) = kawm$ where $w = \exp t_1X_1$ and $m \in G_X$.

As in Proposition 4.8 we have to calculate the pullback under ψ of the Haar measure dx of G. The map ψ is an isomorphism so for any $f \in C_c^\infty(G)$,

$$\int_G f(x)\,dx = \int_{K/L_0}\int_A\int_{\mathbf{R}}\int_{G_X} f(kawm)J_{k,a,w,m}\,dk\,da\,dt_1\,dm$$

where J is the Jacobian. We obtain the formula

$$D\psi_{k,a,w,m}(Z_1, s_2, s_1, Z_2) = L^*_{kawm}\big(\mathrm{Ad}(m^{-1}w^{-1}a^{-1})Z_1 + s_2\mathrm{Ad}(m^{-1}w^{-1})H + s_1\mathrm{Ad}(m^{-1})X_1 + Z_2\big).$$

By using the fact that G_X is unimodular the same argument as in Proposition 4.8 implies

$$J_{k,a,w,m} = \big|\det\big(\mathrm{Ad}\, a^{-1}Z_1 + s_2H + s_1X_1 + \mathrm{Ad}\, wZ_2\big)\big|.$$

We observe that

$$\mathrm{Ad}\, w = e^{t_1 \mathrm{ad}\, X_1} = I + t_1\mathrm{ad}\, X_1 + t_1^2(\mathrm{ad}\, X_1)^2/2 + \cdots$$

so in view of the fact that X_1 is nilpotent, we have

(1) $\mathrm{Ad}\, w|_{\mathfrak{L}_0} = \mathrm{id}$,

(2) $\mathrm{Ad}\, wZ_2 = Z_2 + t_1[Z_2, X_1] +$ higher order terms $(Z_2 \in \mathfrak{L}/\mathfrak{L}_0)$,

(3) $\mathrm{Ad}\, wZ_2 = Z_2 +$ higher order terms if $Z_2 \in \mathfrak{n}_0$.

Then $\mathrm{Ad}\, w$ maps $\mathfrak{L} + \mathfrak{n}_0$ into $\mathfrak{L} + \mathfrak{n}$ and is given by upper triangular matrices.

On the other hand $\mathfrak{k}$ has a basis formed by vectors $Z + \tau Z$ where $Z \in \mathfrak{n}$ and vectors $Z' \in \mathrm{Cent}_{\mathfrak{k}}H$. Also the map $Z_2 \to [Z_2, X_1]$ is onto the orthogonal complement of X_1 in $\mathfrak{n}_1$. By putting all these facts together we can calculate the determinant of $D\psi$. It equals $t_1^{\dim \mathfrak{n}_1 - 1}e^{2\rho(\log a)}$ where the power of t_1 comes from relation (2) and $e^{2\rho(\log a)}$ comes from the action $\mathrm{Ad}\, a^{-1}$ on $\mathfrak{k}$.

Finally the formula in the proposition is a consequence of the normalization of the measures so that $dx = d_{G/G_X}\dot{x}d_X x^*$.

6. The relationship between the unipotent and semisimple integrals. We use the same notation as before. Let $G_{Z_0} = \mathrm{Cent}_G Z_0$ and $G_X = \mathrm{Cent}_G X$. The goal of this section is to prove the formula in Theorem 6.5.

By Remark 3.3 we may assume that $X, H, Y, W \in \mathrm{su}(2, 1) \subseteq g$. According to the nature of the eigenvalues of ad H there are two cases.

Case I. ad H has even eigenvalues only. This is a special case of some results on even nilpotent elements of R. Rao. We give an independent argument. We may set

$$X = \begin{bmatrix} 0 & -i & -i \\ -i & 0 & 0 \\ +i & 0 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & -i & i \\ -i & 0 & 0 \\ -i & 0 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We obtain the following relation

$$Z_0^{\exp t_1 W} = i \begin{bmatrix} 2\sinh 2t_1 & 0 & -2\cosh 2t_1 \\ 0 & 0 & 0 \\ 2\cosh 2t_1 & 0 & -2\sinh 2t_1 \end{bmatrix}.$$

Then

$$Z_0^{\exp t_1 W} = \cosh 2t_1 Z_0 + \sinh 2t_1 X_0 + \sinh 2t_1 (X_2 - \tau X_2), \tag{1}$$

where

$$X_0 = \begin{bmatrix} 2i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{bmatrix}$$

is in g_0 and

$$X_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -i/2 & i/2 \\ 0 & -i/2 & i/2 \end{bmatrix}$$

is in g_4. Then

$$tZ_0^{\exp t_1 W} = \left[\cosh 2t_1 (X - t^2 Y) + t \sinh 2t_1 X_0 + \sinh 2t_1 \left(\frac{1}{t} X_2 - t^3 \tau X_2\right)\right]^{\exp \theta H} \tag{2}$$

where $\theta = \frac{1}{2} \log t$.

In the notation of Proposition 4.8 we consider

$$\int_0^\infty \int_0^\infty g(t_1, t_2) f(tZ_0^{aw})\, dt_1\, dt_2 \tag{3}$$

and by using relations (1) and (2) we make the change of variables $t_2^1 = t_2 + \theta$ and $t_1^1 = t_1/t$. The integral transforms into

$$\int_0^\infty \int_{(1/2)\log t}^\infty tg(tt_1, t_2 - \theta) f(t, t_1, t_2)\, dt_1\, dt_2 \tag{4}$$

where

$$f(t, t_1, t_2) = f\left(\frac{1}{t} \sinh 2tt_1 X_2 + Z_{t,t_1} \right)^{\exp t_2 H}. \tag{5}$$

The term Z_{t,t_1} is equal to

$$Z_{t,t_1} = \cosh 2tt_1 X + t \sinh 2tt_1 X_0 - t^2 \cosh 2tt_1 Y - t^3 \sinh 2tt_1 \tau X_2.$$

LEMMA 6.1. *The function $f(t, t_1, t_2)$ has compact support in t_1 and $t_2 \geqslant 0$ independent of t.*

PROOF. From formula (2) we see that Z_{t,t_1} is orthogonal to X_2 with respect to $-B_\tau$. Then

$$\left\| \left(\frac{1}{t} \sinh 2tt_1 X_2 + Z_{t,t_1} \right)^{\exp t_2 H} \right\|^2 \geqslant e^{8t_2} \frac{1}{t^2} (\sinh 2tt_1)^2 \|X_2\|^2. \tag{6}$$

Since $((tt_1)^{-1} \sinh 2tt_1)^2 \geqslant C > 0$ for any t and t_1 we can write

$$\left\| \left(\frac{1}{t} \sinh 2tt_1 X_2 + Z_{t,t_1} \right)^{\exp t_2 H} \right\|^2 \geqslant Ct_1^2 e^{8t_2} \tag{7}$$

for any t. Since f has compact support, the proof is complete.

We rewrite the integral in (3) as

$$t\int_0^\infty \int_{(1/2)\log t}^0 g(tt_1, t_2 - \theta) f(t, t_1, t_2)\, dt_1\, dt_2$$
$$+ t\int_0^\infty \int_0^\infty g(tt_1, t_2 - \theta) f(t, t_1, t_2)\, dt_1\, dt_2. \tag{8}$$

LEMMA 6.2. *Let $r = \dim \operatorname{Cent} X$ and $n - \dim \mathfrak{g}$. Then*

$$2\rho(H) = n - r + 2(\dim \mathfrak{L} - \dim \mathfrak{L}_0 + 1) = n - r + 2 \dim \mathfrak{p}_1.$$

PROOF. We have $2\rho(H) = 2 \dim \mathfrak{g}_2 + 4 \dim \mathfrak{g}_4$. Since the map ad W: $\mathfrak{L} \to \mathfrak{p}_1$ is onto the orthogonal complement of W we get

$$\dim \mathfrak{L} - \dim \mathfrak{L}_0 + 1 = \dim \mathfrak{p}_1.$$

Also

$$n = \dim g_0 + 2(\dim g_2 + \dim g_4),$$
$$r = \dim \mathfrak{L} + \dim g_2 - \dim \mathfrak{p}_1 + \dim g_4 = \dim g_0 \quad \text{and}$$
$$\dim g_4 = \dim \mathfrak{p}_1$$

since ad X: $g_2 \to g_4$ is onto and ad H is even. Thus

$$\begin{aligned} n - r + 2\dim \mathfrak{p}_1 &= \dim g_0 + 2(\dim g_2 + \dim g_4) - \dim g_0 + 2\dim \mathfrak{p}_1 \\ &= 2\dim g_2 + 4\dim g_4 \end{aligned}$$

and the proof of the lemma is complete.

PROPOSITION 6.3. *For any $f \in C_c(g)$,*

$$\lim_{t\to 0^+} t^{(n-r)/2}\int_{G/G_{Z_0}} f(tZ_0^x)\,dx = c_X\int_{G/G_X} f(X^x)\,dx$$

where c_X depends only on the conjugacy class of X.

PROOF. We note that in formula (3) $g \neq 0$ so we may replace g by det ϕ where ϕ has the expression given in Proposition 4.8. The same holds for formula (8). We note that

$$\lim_{t\to 0}\left(\frac{1}{t}\sinh 2tt_1X_2 + \cosh 2tt_1X + Z'_{t,t_1}\right) = X + 2t_1X_2 = X^{\exp t_1X_1}$$

where X_1 is defined by $X_1 - \tau X_1 = W$, $X_1 \in g_2$ and

$$Z'_{t,t_1} = Z_{t,t_1} - \cosh 2tt_1X.$$

We then have to calculate $\lim_{t\to 0^+} t^{(n-r)/2+1} g(tt_1, t_2 - \theta)$. We choose a basis of $\mathfrak{k}/\mathfrak{L}_0$ consisting of $X_i + \tau X_i$ and Y_j, where $X_i \in g_i$ with $i > 0$ and $Y_j \in \mathfrak{L}$ orthogonal to $\mathfrak{L}_0$. For $\mathfrak{g}_Z$ we choose an orthogonal basis consisting of Y_j and Z_k where Y_j are as before. Then

$$\begin{aligned} &t^{(n-r)/2+1} g(tt_1, t_2 - \theta) \\ &\quad = t^{(n-r)/2+1}\det\left(e^{-it_2}t^{i/2}X_i + e^{it_2}t^{-i/2}\tau X_i, Y_j, H, W, \operatorname{Ad} wY_j, \operatorname{Ad} wZ_k\right) \\ &\quad = t_1^{\dim \mathfrak{p}_1 - 1}\det\left(e^{-it_2}t^{i}X_i + e^{it_2}\tau X_i, Y_j, H, W, \frac{\operatorname{Ad} wY_j - Y_j}{tt_1}, \operatorname{Ad} wZ_k\right) \end{aligned}$$

by using the formula for $n - r$ in Lemma 6.2. Since $w = \exp tt_1 W$,

$$\lim_{t\to 0^+}\frac{\operatorname{Ad} wY_j - Y_j}{tt_1} = [W, Y_j].$$

Thus

$$\begin{aligned} &\lim_{t\to 0} t^{(n-r)/2+1} g(tt_1, t_2 - \theta) \\ &\quad = t_1^{\dim \mathfrak{p}_1 - 1} e^{2t_2\rho(H)}\det\left(\tau X_i, Y_j, H, W, [W, Y_j], Z_k\right) \\ &\quad = \text{const}\; t_1^{\dim \mathfrak{p}_1 - 1} e^{2t_2\rho(H)}. \end{aligned}$$

Since $f(t, t_1, t_2)$ has compact support in t_1 and $t_2 > 0$ independent of t, the second integral in (8) converges to

$$\int_0^\infty \int_0^\infty e^{2\rho(\log a)} t_1^{\dim p_1 - 1} f(X^{\exp t_1 X_1}) \, dt_1 \, dt_2$$

by the dominated convergence theorem. For the term for which $t_2 < 0$, we expand the determinant into powers of e^{t_2}. For positive powers of e^{t_2} we get terms of the form $e^{jt_2} t^{(2\rho(H)-j)/2}$ where $0 < j \leqslant 2\rho(H)$. All these terms are absolutely integrable on $(-\infty, 0)$ so we may take the limit inside the integral sign. All terms that have $j < 2\rho(H)$ converge to 0 while for $j = 2\rho(H)$ we get

$$e^{2t_2\rho(H)} t^{\dim p_1 - 1} \det\left(\tau X_i, Y_j, H, W, [W, Y_j], Z_k\right).$$

For strictly negative powers of e^{t_2} we make a change of variables $u = t_2 - \log t$. The integral becomes a sum of terms

$$\int_0^\infty \int_{-(1/2)\log t}^\infty t_1^{\dim p_1 - 1} e^{ju} t^{(2\rho(H)+j)/2}$$
$$\cdot \det\left(\tilde{X}_i, Y_j, H, W, \frac{\operatorname{Ad} wY_j - Y_j}{tt_1}, \operatorname{Ad} wZ_k\right) \tilde{f}\,(t, t_1, t_2) \, dt_1 \, dt_2$$

where $\tilde{X}_i$ are either X_i or τX_i. Since $j < 0$, e^{ju} is absolutely integrable and $\tilde{f}(t, t_1, t_2)$ is bounded in the interval $(0, \infty)$, all the terms converge to 0 (we note that $-\frac{1}{2}\log t \to +\infty$).

Finally the term for $j = 0$ can be estimated by

$$t^{\rho(H)} \int_0^\infty \int_{-(1/2)\log t}^0 f(t, t_1, t_2) \, dt_1 \leqslant C t^{\rho(H)} |\log t|.$$

Since $\rho(H) \geqslant 1$ this term also converges to 0.

Putting all these facts together we get the formula

$$\lim_{t \to 0^+} t^{(n-r)/2} \int_0^\infty \int_0^\infty g(t_1, t_2) f(tZ_0^{aw}) \, dt_1 \, dt_2$$
$$= \int_0^\infty \int_{-\infty}^\infty e^{2\rho(\log a)} t_1^{\dim p_1 - 1} f(X^{\exp t_1 X_1}) \, dt_1 \, dt_2. \tag{9}$$

In Proposition 4.8 we can split the right-hand side of the formula into a sum of two integrals each of which converges to the corresponding expression to the right-hand side of (9). We thus get

$$\lim_{t\to 0^+} t^{(n-r)/2}\int_{G/G_{Z_0}} f(tZ_0^x)\,dx$$

$$= \frac{1}{c_{Z_0}}\int_{K/L_0}\int_A\int_{\mathbf{R}} e^{2\rho(\log a)}t_1^{\dim p_1 - 1}f(X^{kaw})\,dt_1\,dt_2$$

$$= c_X\int_{G/G_X} f(X^x)\,dx$$

and $c_X = \text{const}/c_{Z_0}$ and Z_0 depend only on the conjugacy class of X by Proposition 3.1.

This completes Case I.

Case II. ad H has even as well as odd eigenvalues. We can identify X, H, Y, $W \in \mathrm{su}(2,1) \subseteq g$ with

$$X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & i/2 & i/2 \\ 0 & -i/2 & -i/2 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & +i/2 & -i/2 \\ 0 & +i/2 & -i/2 \end{bmatrix},$$

$$H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

and

$$W = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then

$$Z_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}.$$

Let $w = \exp t_1 W$ and $a = \exp t_2 H$. Then we have the relation

$$Z_0^w = i\begin{bmatrix} \sinh^2 t_1 & 0 & -\sinh t_1 \cosh t_1 \\ 0 & 1 & 0 \\ \sinh t_1 \cosh t_1 & 0 & -\cosh t_1 \end{bmatrix}. \tag{10}$$

Then

$$tZ_0^w = \left[\left(\frac{1+\cosh^2 t_1}{2}\right)(X - t^2 Y) + Z_{t,t_1}\right]^{\exp\theta H} \tag{11}$$

where $\theta = \frac{1}{2}\log t$ and

$$Z_{t,t_1} = -\tfrac{1}{2}\sinh 2t_1\left(t^{1/2}X_1 + t\tau X_1\right) + \tfrac{1}{2}t\sinh^2 t_1 X_0. \tag{12}$$

Here $X_1 \in g_1$ and $X_0 \in g_0$ are given by

$$X_1 = \begin{bmatrix} 0 & i/2 & i/2 \\ i/2 & 0 & 0 \\ -i/2 & 0 & 0 \end{bmatrix}, \qquad X_0 = \begin{bmatrix} 2i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{bmatrix}.$$

From formula (12) we see that $Z_{t,t_1} \to 0$ when $t \to 0$.

In the notation of Proposition 4.8 we consider

$$\int_0^\infty \int_0^\infty g(t_1, t_2) f(tZ_0^{aw})\, dt_1\, dt_2. \tag{13}$$

By using relations (10) and (11) we make the change of variables $t_2' = t_2 + \theta$. The integral in (12) becomes

$$\int_0^\infty \int_{(1/2)\log t}^\infty g(t_1, t_2 - \theta) f(t, t_1, t_2)\, dt_1\, dt_2 \tag{14}$$

where

$$f(t, t_1, t_2) = f\left[\left(\frac{1 + \cosh^2 t_1}{2}(X - t^2 Y) + Z_{t,t_1}\right)^a\right]. \tag{15}$$

Lemma 6.4. *The function $f(t, t_1, t_2)$ has compact support in t_1 and $t_2 \geqslant 0$ independent of t.*

Proof. Since X is orthogonal to Y and Z_{t,t_1} we have the relation

$$\left\| \left\{ \left(\frac{1 + \cosh^2 t_1}{2} \right)(X - t^2 Y) + Z_{t,t_1} \right\}^{\exp t_2 H} \right\|^2 \geqslant \left(\frac{1 + \cosh^2 t_1}{2} \right)^2 e^{4t_2} \|X\|^2. \tag{16}$$

The statement in the lemma now follows from inequality (16) and the fact that f has compact support.

We rewrite the integral in (14) as

$$\int_0^\infty \int_{(1/2)\log t}^0 g(t_1, t_2 - \theta) f(t, t_1, t_2)\, dt_1\, dt_2 + \int_0^\infty \int_0^\infty g(t_1, t_2 - \theta) f(t, t_1, t_2)\, dt_1\, dt_2. \tag{17}$$

Lemma 6.5. *Let $r = \dim \mathrm{Cent}_{\mathfrak{g}} X$ and $n = \dim \mathfrak{g}$. Then*

$$2\rho(H) = n - r. \tag{18}$$

PROOF. The lemma is an immediate consequence of the relations

$$2\rho(H) = \dim g_1 + 2\dim g_2,$$
$$n = \dim g_0 + 2\dim g_1 + 2\dim g_2,$$
$$r = \dim \mathfrak{L} + \dim g_1 + \dim g_2, \quad \text{and}$$
$$\dim g_0 - \dim \mathfrak{L} = \dim g_2.$$

PROPOSITION 6.6. *Let* $f \in C_c^\infty(g)$. *Then*

$$\lim_{t\to 0^+} t^{(n-r)/2}\int_{G/G_{Z_0}} f(tZ_0^x)\,dx = c_X\int_{G/G_X} f(X^x)\,dx \tag{19}$$

and c_X *depends only on the conjugacy class of* X *in* g.

PROOF. We note that in formula (17) $g \neq 0$ over the range of the integral as we may take $\pm\det\phi$ in the notation of Proposition 4.8. From formula (11) we conclude that

$$\lim_{t\to 0}\left\{\left(\frac{1+\cosh^2 t_1}{2}\right)(X - t^2Y) + Z_{t,t_1}\right\}^a = \frac{1+\cosh^2 t_1}{2}X^a.$$

As in Case I we select a basis of $\mathfrak{k}/\mathfrak{L}_0$ consisting of vectors $X_i + \tau X_i$ and Y_j where $X_i \in g_i$ with $i > 0$ and $Y_j \in \mathfrak{L}$ orthogonal to $\mathfrak{L}_0$. For $\mathfrak{g}_Z$ we select a basis consisting of Y_j and Z_k. Then

$$\begin{aligned} t^{(n-r)/2} g(t_1, t_2 - \theta) &= t^{(n-r)/2}\det\left(e^{-it_2}t^{i/2}X_i + e^{it_2}t^{-i/2}\tau X_i, Y_j, H, W, \operatorname{Ad} wY_j, \operatorname{Ad} wZ_k\right) \\ &= \det\left(e^{-it_2}t^iX_i + e^{it_2}\tau X_i, Y_j, H, W, \operatorname{Ad} wY_j, \operatorname{Ad} wZ_k\right) \end{aligned} \tag{20}$$

by multiplying the t inside the determinant. Using formula (18) this expression converges to

$$e^{2\rho(\log a)}c(t_1) = e^{2\rho(\log a)}\det(\tau X_i, Y_j, H, W, \operatorname{Ad} wY_j, \operatorname{Ad} wZ_k). \tag{21}$$

Due to Lemma 6.4 we can use the dominated convergence for the second term in (17). Its limit is

$$\int_0^\infty\int_0^\infty e^{2\rho(\log a)}c(t_1)f\left(\frac{1+\cosh^2 t_1}{2}X^a\right)dt_1\,dt_2.$$

For the first term in (17) we expand in powers of e^{t_2}. For $0 < j \leqslant 2\rho(H)$ we get $e^{jt_2}t^{(2\rho(H)-j)/2}\det(\tilde{X}_i, Y_j, H, W, \operatorname{ad} wY_j, \operatorname{Ad} wZ_k)$ where $\tilde{X}_i$ is either X_i or τX_i. Then $e^{jt_2}f(t, t_1, t_2)$ is bounded by an absolutely integrable function for any t_1 and t_2 so we can use the bounded convergence theorem for each term. Only the term for $j = 2\rho(H)$ contributes anything. We get

$$\int_0^\infty \int_{-\infty}^0 e^{2\rho(\log a)} c(t_1) f\left(\frac{1 + \cosh^2 t_1}{2} X^a \right) dt_1\, dt_2.$$

For $-2\rho(H) \leqslant j < 0$ we make the change of variables $u = t_2 - \log t$. The integrals transform into

$$\int_0^\infty \int_{-(1/2)\log t}^\infty e^{ju} t^{(2\rho(H)+j)/2}$$

$$\cdot \det\left(\tilde{X}_i, Y_j, H, W, \operatorname{Ad} wY_j, \operatorname{Ad} Z_k\right) \tilde{f}\,(t, t_1, u)\, dt_1\, dt_2$$

where $\tilde{X}_i$ is either X_i or τX_i and $\tilde{f}(t, t_1, u) = f(t, t_1, u + \log t)$ is a bounded function in (t, t_1, u). We can apply the bounded convergence theorem. Since $-\frac{1}{2}\log t \to +\infty$ the limit of any of these integrals is 0. Remains to consider the case $j = 0$. The same argument as in Case I applies. We have obtained the formula

$$\lim_{t \to 0^+} t^{(n-r)/2} \int_0^\infty \int_0^\infty g(t_1, t_2) f(tZ_0^{aw})\, dt_1\, dt_2$$

$$= \int_0^\infty \int_{-\infty}^\infty e^{2\rho(\log a)} c(t_1) f\left(\frac{1 + \cosh^2 t_1}{2} X \right) dt_1\, dt_2. \quad (22)$$

We make the change of variables $t' = t + \frac{1}{2} \log((1 + \cosh^2 t_1)/2)$ in the right-hand side of (22) to get

$$\int_0^\infty \frac{c(t_1)}{\left((1 + \cosh^2 t_1)/2\right)^{\rho(H)}}\, dt_1 \int_{-\infty}^\infty e^{2\rho(\log a)} f(X^a)\, dt_2. \quad (23)$$

The proof of the proposition is completed in the same manner as in Case I. We now put these results together in the following theorem.

THEOREM 6.7. *Let $f \in C_c^\infty(G)$. Then*

$$\lim_{t \to 0^+} t^{(n-r)/2} \int_{G/G_{Z_0}} f(z_t^x)\, dx = c_X \int_{G/G_u} f(u^x)\, dx \quad (24)$$

where $u = \exp X$, $z_t = \exp tZ_0$, $r = \dim \operatorname{Cent}_{\mathfrak{g}} X$ and $\{X, H, Y\}$ is an arbitrary Lie triple.

PROOF. Let $\mathcal{O} = \{Z \in \mathfrak{g}\colon \operatorname{ad} Z$ has eigenvalues λ with $|\operatorname{Im} \lambda| < \pi\}$. It is well known that the map $\exp\colon \mathcal{O} \to G$ is a diffeomorphism onto an open set $\mathcal{P}$. If $u = \exp X$ and $z_t = \exp tZ_0$ then u and z_t are contained in $\mathcal{P}$ for t small enough. In addition, $\mathcal{O}$ and $\mathcal{P}$ are invariant by the adjoint action so the orbits of u and z_t are contained in $\mathcal{P}$. Without loss of generality we may assume that $\operatorname{supp} f \subseteq \mathcal{P}$. Let $F = f \circ \exp$. If F does not have compact support we can always find F_1 with support in a set $\mathcal{O}_f$ such that

(1) $\mathcal{O}_f \subseteq \mathcal{O}$ and supp $f \subseteq \exp \mathcal{O}_f$,
(2) $F_1 \in C_c^\infty(g)$,
(3) $F_1|_{\mathcal{O}_f} = f \circ \exp$.

By applying Propositions 6.3 or 6.6 to F_1 and $\{X, H, Y\}$ we get

$$\lim_{t\to 0^+} t^{(n-r)/2}\int_{G/G_{Z_0}} F_1(tZ_0^x)\,dx = c_X\int_{G/G_X} F_1(X^x)\,dx.$$

We now note that $G_X = G_u$ and $G_{Z_0} = G_{z_t}$ for t small enough. The first statement follows from Lemma 5.1. If t is small enough the eigenvalues of tZ_0 have absolute value less than π. Then $\mathrm{Cent}_{G_c} tZ_0 = \mathrm{Cent}_{G_c} z_t$ is a connected group. Thus we can write

$$\lim_{t\to 0^+} t^{(n-r)/2}\int_{G/G_{Z_0}} f(z_t^x)\,dx = c_X\int_{G/G_u} f(u^x)\,dx$$

and the proof is complete.

We rewrite this formula in terms of the distribution Φ_f^B defined in §1.

COROLLARY 6.8. *Under the assumptions of Theorem* 6.7 *let* $p = r - s$ *where* $s = \dim \mathrm{Cent}_g H$. *Then*

$$\left(\frac{d}{dt}\right)^p\Big|_{t=0} \Phi_f^B\left(z_t;\bar{\omega}_{Z_0}\right) = \Phi_f^B\left(1;\bar{\omega}_{Z_0}Z_0^p\right) = c_u\int_{G/G_u} f(u^x)\,dx.$$

PROOF. Fix a connected component $B^+ \subseteq B'$ such that $z_t \in \mathrm{cl}(B^+)$. By the work of Harish-Chandra it is known that $\Phi_f^B(z_t;\bar{\omega}_{Z_0})$ is well defined independent of B^+. By Lemma 2.7 applied to $\gamma = z_t$ we get

$$\Phi_f^B\left(z_t;\bar{\omega}_{Z_0}\right) = k_t\int_{G/G_{Z_0}} f(z_t^x)\,dx$$

where

$$k_t = d_{Z_0}\xi_{-\rho}(z_t)\prod_{\beta\in P_{g/Z_0}}\left(\xi_\beta(z_t) - 1\right).$$

Then

$$\lim_{t\to 0^+} t^{(n-r)/2}k_t^{-1}\Phi_f^B\left(z_t;\bar{\omega}_{Z_0}\right) = c_X\int_{G/G_u} f(u^x)\,dx$$

by Theorem 6.7. Since $[P_{g/Z_0}] = (n-s)/2$ we can write

$$\lim_{t\to 0^+} t^{(n-s)/2}k_t^{-1} = \frac{1}{d_{Z_0}}\prod_{\beta\in P_{g/Z_0}}\beta(Z_0)^{-1}.$$

Thus, since Φ_f^B is a Schwartz function, we can write

$$\lim_{t\to 0^+} t^{-p}\Phi_f^B\left(z_t;\bar{\omega}_{Z_0}\right) = \mathrm{const}\int_{G/G_u} f(u^x)\,dx.$$

The proof will be complete if we show that

$$\lim_{t\to 0^+}(p!)t^{-p}\Phi_f^B(z_t;\bar{\omega}_{Z_0}) = \lim_{t\to 0^+}\Phi_f^B(z_t;\bar{\omega}_{Z_0}Z_0^p)$$

$$= \Phi_f^B(1;\bar{\omega}_{Z_0}Z_0^p) = \left(\frac{d}{dt}\right)^p\Big|_{t=0}\Phi_f^B(z_t;\bar{\omega}_{Z_0}).$$

This is a consequence of the elementary Lemma 6.9. Putting all the coefficients together we get the following expression for c_u.

$$c_u = (p!)d_{Z_0}c_X \prod_{\beta\in P_{\mathfrak{g}/Z_0}}(\beta, Z_0). \tag{25}$$

LEMMA 6.9. *Let $F \in C^\infty(0, \varepsilon]$ be such that for any integer k, D_t^kF has a limit when $t \to 0$ ($D_t = d/dt$). Then if $\lim_{t\to 0^+}t^{-p}F(t)$ exists, it equals $(p!)^{-1}D_t^pF(0)$.*

PROOF. Extend F to a C^{p+2} function on $(-\varepsilon, \varepsilon)$. The Taylor series expansion of F is

$$F(t) = F(0) + \frac{F'(0)}{1!}t + \cdots + \frac{F^{(p)}(0)}{p!}t^p + O(t^{p+1}).$$

If $\lim_{t\to 0}t^{-p}F(t)$ exists, then it is clear that

$$F(0) = \cdots = F^{(p-1)}(0) = 0 \text{ and } \lim_{t\to 0^+}t^{-p}F(t) = (p!)^{-1}D_t^pF(0).$$

7. The Fourier transform of the unipotent integral. In this section we calculate the Fourier transform of the distribution $T_u(f)$ for $u \in G$. This will be done by using the results in §6 and [**13**].

We use the notation in §2. In order to use the results in [**12**] we explain the relevant notation used there. Let B be the compact CSG. Then $A_K = Z(A)A_K^0$ where A_K^0 is the connected component of A_K and $Z(A) = \{1, v\}$ where $v = \exp(\sqrt{-1}\,\pi H_\alpha) = \exp(\pi(E_\alpha - E_{-\alpha}))$.

We let $\mathfrak{b}_2 = \mathbf{R}(E_\alpha - E_{-\alpha})$ and $\mathfrak{b}_1$ its orthogonal complement in $\mathfrak{b}$. Let B_1 and B_2 be the corresponding subgroups. Let now $Z \in \mathfrak{b}$ and $z = \exp Z$. Set P_Z as in §2, Lemma 2.7. Let $W_z(G, B)$ be the subgroup of $W(G, B)$ generated by the compact roots in P_Z. Choose $w_1 = 1, \ldots, w_n$ such that

$$W(G, B) = \bigcup_{i=1}^{N} W_z(G, B)w_i \quad \text{(disjoint union)}.$$

Then if $w \in W(G, B)$, $w = w_2w_i$. Moreover

$$w_i^{-1}z = z_1(w_i)z_2(w_i), \qquad z_1 \in B_1, z_2 \in B_2.$$

Since this decomposition is unique only up to $\{1, v\}$ we normalize z_2 such that

$$z_2(w_i) = \exp\theta_{w_i}(E_\alpha - E_{-\alpha}), \qquad -\pi/2 \leqslant \theta_{w_i} < \pi/2.$$

We will take $Z = tZ_0$ for t small enough so that P_Z and P_z are identical.

We define the functions

$$F_\chi^{\pm}(w\colon \nu\colon z) = \overline{w\chi(z)}\left\{e^{\mp\nu\pi/2}\left(w\xi_{\tilde{\alpha}}(z)\right)^{\sqrt{-1}\,\nu/2} - e^{\pm\nu\pi/2}\left(w\xi_{\tilde{\alpha}}(z)\right)^{-\sqrt{-1}\,\nu/2}\right\}$$

$$= \left\{e^{\mp\nu\pi/2}\exp\left[\left(\log\bar{\chi} + \frac{\sqrt{-1}\,\nu}{2}\tilde{\alpha}, w^{-1}Z\right)\right]\right.$$

$$\left. - e^{\pm\nu\pi/2}\exp\left[\left(\log\bar{\chi} - \frac{\sqrt{-1}\,\nu}{2}\tilde{\alpha}, w^{-1}Z\right)\right]\right\},$$

$$G_\chi^{\pm}(w\colon \nu\colon z) = \overline{\pm w\chi(z)}\left\{e^{\mp\nu\pi/2}\left(w\xi_{\tilde{\alpha}}(z)\right)^{\sqrt{-1}\,\nu/2}\right.$$

$$\left. + e^{\pm\nu\pi/2}\left(w\xi_{\tilde{\alpha}}(z)\right)^{-\sqrt{-1}\,\nu/2}\right\}$$

$$= \pm\left\{e^{\mp\nu\pi/2}\left(\log\bar{\chi} + \frac{\sqrt{-1}\,\nu}{2}\tilde{\alpha}, w^{-1}Z\right)\right.$$

$$\left. + e^{\pm\nu\pi/2}\left(\log\bar{\chi} - \frac{\sqrt{-1}\,\nu}{2}\tilde{\alpha}, w^{-1}Z\right)\right\}.$$

Let $r = [P]$, $r_z = [P_z]$ and $r_I = [P_I]$. We also define $\hat{A}_K^{\pm} = \{\chi\colon \chi(v) = \pm 1\}$ and $\varepsilon(\chi) = \operatorname{sgn}\prod_{\alpha\in P_I}(\log\chi, \alpha)$.

We are now ready to state the main result of this section.

THEOREM 7.1. *Let $u = \exp X$ be unipotent. Then the distribution $c_u T_u(f)$ equals*

$$(-1)^{r+r_{Z_0}}\sum_{\mu\in L_B}\left|\prod_{\beta\in P_{Z_0}}(\mu, \beta)\right|\left[\overline{(\mu, Z_0)}\right]^p\Theta_\mu(f)$$

$$+ (\sqrt{-1}/4)(-1)^{r_I}[W_{Z_0}(G, B)]/[W(G, A)]\sum_{\chi\in\hat{A}_K^{+}}\varepsilon(\chi)$$

$$\times\left\{\sum_{\substack{w_i \\ 0<\theta_{w_i}<\pi/2}}\det(w_i)\int_{-\infty}^{+\infty}T^{(\chi,\nu)}(f)\left[F_\chi^{+}(w_i\colon \nu\colon 1; \bar{\omega}_{Z_0}Z_0^\beta)/\sinh(\nu\pi/2)\right]d\nu\right.$$

$$\left. + \sum_{\substack{w_i \\ -\pi/2<\theta_{w_i}<0}}\det(w_i)\int_{-\infty}^{+\infty}T^{(\chi,\nu)}(f)\left[F_\chi^{-}(w_i\colon \nu\colon 1; \bar{\omega}_{Z_0}Z_0^\beta)/\sinh(\nu\pi/2)\right]d\nu\right\}$$

$$+ (\sqrt{-1}/4)(-1)^{r_I}[W_{Z_0}(G; B)]/[W(G, A)]\sum_{\chi\in\hat{A}_K^{-}}\varepsilon(\chi)$$

$$\times\left\{\sum_{\substack{w_i \\ 0<\theta_{w_i}<\pi/2}}\det(w_i)\int_{-\infty}^{+\infty}T^{(\chi,\nu)}(f)\left[G_\chi^{+}(w_i\colon \nu\colon 1; \bar{\omega}_{Z_0}Z_0^\beta)/\cosh(\nu\pi/2)\right]d\nu\right.$$

$$\left. + \sum_{\substack{w_i \\ -\pi/2<\theta_{w_i}<0}}\det(w_i)\int_{-\infty}^{+\infty}T^{(\chi,\nu)}(f)\left[G_\chi^{-}(w_i\colon \nu\colon 1; \bar{\omega}_{Z_0}Z_0^\beta)/\cosh(\nu\pi/2)\right]d\nu\right\}.$$

PROOF. This formula follows immediately by applying Theorem 6.7 to Theorems 3.19 and 5.21 in [13], substituting $y = z_t = \exp tZ_0$ and differentiating. Lemma 68 on p. 315 in [5c] for the discrete series part and the form of $T^{(\chi,\nu)}$ given in [13] can be used to justify term by term differentiation.

The formulas for $F_\chi^\pm$ and $G_\chi^\pm$ occurring in the statement are given below

$$F_\chi^\pm\left(w\colon \nu\colon 1;\, \bar{\omega}_{Z_0} Z_0^p\right) = \left\{c_1 e^{\mp \nu\pi/2}\left[\left(\overline{\log \chi} + \frac{\sqrt{-1}\,\nu}{2}\tilde{\alpha},\, w^{-1}Z_0\right)\right]^p - c_2 e^{\pm \nu\pi/2}\left[\left(\log \bar{\chi} - \frac{\sqrt{-1}\,\nu}{2}\tilde{\alpha},\, w^{-1}Z_0\right)\right]^p\right\}$$

$$G_\chi^\pm\left(w\colon \nu\colon 1;\, \bar{\omega}_{Z_0} Z_0^p\right) = \left\{c_1 e^{\mp \nu\pi/2}\left[\left(\log \bar{\chi} + \frac{\sqrt{-1}\,\nu}{2}\tilde{\alpha},\, w^{-1}Z_0\right)\right]^p + c_2 e^{\pm \nu\pi/2}\left[\left(\log \bar{\chi} - \frac{\sqrt{-1}\,\nu}{2}\tilde{\alpha},\, w^{-1}Z_0\right)\right]^p\right\},$$

where

$$c_1 = \prod_{\beta \in P_{Z_0}} \left(w\left(\log \bar{\chi} + \frac{\sqrt{-1}\,\nu}{2}\tilde{\alpha}\right), \beta\right),$$

$$c_2 = \prod_{\beta \in P_{Z_0}} \left(w\left(\log \bar{\chi} - \frac{\sqrt{-1}\,\nu}{2}\tilde{\alpha}\right), \beta\right).$$

REMARK. These formulas become simpler when Z_0 is regular, i.e. $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{R})$ or $\mathfrak{g} = \mathfrak{su}(2, 1)$ as Theorem 3.19 in [13] suggests. Similar formulas for $u \in G$ arbitrary were calculated in [2] using Jordan decomposition.

8. The constant c_u. In this section we evaluate the constant c_u relevant to the calculations in [10]. In view of this we note that if the discrete series part of the distribution $T_u(f)$ is $\Sigma a(\mu)\theta_\mu(f)$ then up to a constant factor,

$$a(\mu) = (\overline{\mu, Z_0})^p \prod_{\beta \in P_{Z_0}} (\mu, \beta) \Big/ \prod_{\beta \in P_{\mathfrak{g}/Z_0}} (\beta, Z_0).$$

LEMMA 8.1. *Suppose that there is an element $x \in G$ such that $X^x = -X$. Then the discrete series part in Theorem 7.1 is equal to zero.*

PROOF. If $X^x = -X$ then there is $w \in W(G, B)$ such that $Z_0^w = -Z_0$ by the results in §3. Furthermore we must have $\dim \mathfrak{g}_2 > 1$. By the formulas for θ_μ in [12] we know that $\theta_{w\mu} = \varepsilon(w)\theta_\mu$. Both θ_μ and $\theta_{w\mu}$ occur in the sum. We compute $a(w\mu)$. We have

$$a(w\mu) = (\overline{w\mu, Z_0})^p \prod_{\beta \in P_{Z_0}} (w\mu, \beta) / \prod_{\beta \in P_{g/Z_0}} (\beta, Z_0)$$

$$= (-1)^p \varepsilon_{Z_0}(w) (\overline{\mu, Z_0})^p \prod_{\beta \in P_{Z_0}} (\mu, \beta) / \prod_{\beta \in P_{g/Z_0}} (\beta, Z_0)$$

$$= (-1)^p \varepsilon_{Z_0}(w) a(\mu)$$

where $\varepsilon_{Z_0}(w)$ is the number of roots in P_{Z_0} mapped into $-P_{Z_0}$ by w. We recall that $p = (r - s)/2$ also equals $\frac{1}{2}$ dim g_1. Assume $p > 0$. We will show that p is even or equivalently that dim $g_1 \equiv 0$ (mod 4). We recall that the complexification of g_1 is generated by root vectors of roots α_1 such that $\alpha_1(H) = 1$. Then $\sigma\alpha_1 \neq \alpha_1$ and $\sigma\alpha_1(H) = 1$. Since dim $g_2 > 1$ let $\beta \in P$ be a root such that $\beta \neq \alpha$ and $\beta(H) = 2$. By Lemma 2.4 either $\beta - \alpha_1$ or $\beta - \sigma\alpha_1$ is a root, say $\beta - \alpha_1 = \beta_1$ is a root. Clearly $\beta - \sigma\alpha_1$ cannot also be a root since otherwise either $(\beta - \alpha_1) - (\beta - \sigma\alpha_1) = \sigma\alpha_1 - \alpha_1$ or $(\beta - \alpha_1) - \sigma(\beta - \sigma\alpha_1) = \beta - \sigma\beta$ would be a root. Thus P_1 splits into disjoint sets of the form $\{\alpha_1, \sigma\alpha_1, \beta_1, \sigma\beta_1\}$. Thus $[P_1] \equiv 0$ (mod 4). Therefore $a(w\mu) = \varepsilon_{Z_0}(w)a(\mu)$. Finally we show that $\varepsilon_{Z_0}(w)\varepsilon(w) = -1$. We may assume that x is such that $X^x = -X$, $Y^x = -Y$ and $H^x = H$. Let $\mathfrak{m} = \text{Cent}_K H$ and $\mathfrak{m}_{Z_0} = \text{Cent}_{\mathfrak{m}} Z_0$. Then $\mathfrak{m}_{Z_0}^x = \mathfrak{m}_{Z_0}$ is a compact subalgebra containing $\mathfrak{a}_K$ as a Cartan subalgebra. Thus we may replace x with $w \in W(G, B)$ with the same properties. In addition, we may consider w as an element of $W(G, J)$ as well. But w maps Z_0 into $-Z_0$ so it maps P_{g/Z_0} into $-P_{g/Z_0}$. Since $[P_{g/Z_0}] = \dim g_1 + \dim g_2$ is odd and the complement of P_{g/Z_0} in P is P_{Z_0} we have $\varepsilon(w) = (-1)\varepsilon_{Z_0}(w)$. This completes the proof of the lemma.

Thus the only cases when the unipotent orbit has a nontrivial discrete series contribution is when dim $g_2 = 1$. Then dim $g_4 = 0$ and the only cases are the groups su$(n, 1)$ for $n \geqslant 1$. The case $n = 1$ is well known. Thus we assume dim $g_1 > 0$. This is Case II in §6. Let $w = \exp t_1 W$ and X, H, Y, W be as in that case. From the proof of Proposition 6.6 we have to calculate $c(t_1)$ defined by (20). Let $Z_1 \in g^- \oplus \mathfrak{L}/\mathfrak{L}_0$ where $g^- = \sum_{i<0} g^i$ and $Z_2 \in \mathfrak{g}_Z$. Then we have to evaluate the determinant of ϕ: $g^- \oplus \mathfrak{L}/\mathfrak{L}_0 \times \mathbf{R}^2 \times \mathfrak{g}_Z \to g$ given by $\phi(Z_1, s_2, s_1, Z_2) = Z_1^{a^{-1}} + s_2 H + s_1 W + Z_2^w$. By choosing a basis as in Case II we get $e^{2\rho(\log a)}\det(\tau X_i, Y_j, H, W, \text{Ad}\, wY_j, \text{Ad}\, wZ_k)$. We note that $\mathfrak{g}_Z = \mathfrak{L} + \mathbf{R}Z_0$ in this case. Also W and H are conjugate in su(2, 1) by some element $x = s_{\tilde{\alpha}_1}$ where $s_{\tilde{\alpha}_1}$ is the reflection in $W(G, B)$ corresponding to a simple root $\alpha_1 \in \Delta(g_c, j_c)$. Suppose $E_{\beta_1 - \alpha_1} \in \mathfrak{L}_c$ is a root vector. Then

$$wE_{\beta_1 - \alpha_1} = wE_{\tilde{\beta}_1 - \tilde{\alpha}_1} = s_{\tilde{\alpha}_1} \exp t_1 H s_{\tilde{\alpha}_1} E_{\tilde{\beta}_1 - \tilde{\alpha}_1} = s_{\alpha_1} \exp t_1 H E_{\tilde{\beta}_1}.$$

Thus it is enough to calculate $\exp t_1 H E_{\tilde{\alpha}_1}$ since $\tilde{\alpha}_1$ and $\tilde{\beta}_1$ generate a subalgebra isomorphic to su(2, 1). We get $\cosh t_1 E_{\tilde{\beta}_1} - \sinh t_1 E_{\sigma\tilde{\beta}_1}$ and $s_{\tilde{\alpha}_1}$ maps them into

$$\cosh t_1 E_{\tilde\beta_1-\tilde\alpha_1} - \sinh t_1 E_{\tilde\beta_1} = \cosh t_1 E_{\beta_1-\alpha_1} - \sinh t_1 (E_{\beta_1} \pm E_{-\sigma\beta_1}).$$

Furthermore, by relations (10) and (11),

$$Z_0^w = \frac{1+\cosh^2 t_1}{2} Z_0 - \tfrac{1}{2}\sinh 2t_1 (X_1 + \tau X_1) + \tfrac{1}{2}\sinh^2 t_1 X_0.$$

We write

$$Z_0^w = \frac{1+\cosh^2 t_1}{2} Z_0 + Z_{t_1}.$$

Substituting into the expression for φ we get

$$e^{2\rho(\log a)}\det\big(\tau X_i, Y_j, H, W, \operatorname{Ad} w\mathfrak{a}_k, \cosh t_1 E_{\beta_1-\alpha_1} - \sinh t_1 (E_{\beta_1} \pm E_{-\sigma\beta_1}), Z_0^w\big).$$

By elementary column operations this reduces to

$$2^{p-1} e^{2\rho(\log a)} (\sinh t_1)^{2p-2} \det_{su(2,1)}$$

where $\det_{su(2,1)}$ is the corresponding determinant for $g = su(2, 1)$. But this can be calculated explicitly and equals $\sinh 2t_1$. Thus $c(t_1) = (\sinh t_1)^{2p-2}\sinh 2t_1$ and the integral to be evaluated is

$$2^p \int_0^\infty \frac{(\sinh t_1)^{2p-2}\sinh 2t_1}{\big((1+\cosh^2 t_1)/2\big)^{\rho(H)}}\, dt_1$$

which is finite since $2p + 2 = 2\rho(H)$. We get by setting $u = \sinh t_1$

$$\int_0^\infty \frac{(\sinh t_1)^{2p-1}\cosh t_1}{\left(1+\dfrac{\sinh^2 t_1}{2}\right)^{p+1}}\, dt_1 = \int_0^\infty \frac{u^{2p-1}}{\left(1+\dfrac{u^2}{2}\right)^{p+1}}\, du$$

$$= 2^{p-1}\int_1^\infty \frac{(v-1)^{p-1}}{v^{p+1}}\, dv = 2^{p-1}(-1)^{p-1}\int_0^1 (1-v)^{p-1} v^4 dv$$

$$= 2^{p-1}(-1)^{p-1} B(p-1, 4)$$

where $v = 1 + u^2/2$ in the third equality. Putting these calculations together we get the following proposition.

PROPOSITION 8.2. *In the case when the discrete series contribution of T_u is not zero, the constant c_u can be expressed as*

$$c_u = 2^{2p-1}(-1)^{p-1} B(p-1, 4)\cdot c_{Z_0}^{-1}\cdot d_{Z_0}\cdot (p!) \prod_{\beta\in P_{g/z_0}} (\beta, Z_0)$$

where c_{Z_0} *is defined in Proposition* 4.8 *and* d_{Z_0} *in Lemma* 2.7.

PROOF. The statement follows easily from the previous calculation and Corollary 6.8.

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DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139

Current address: School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540